πgβ-Normal Spaces in Topological Spaces

Hamant Kumar[1], Umesh Chand[2], Rajpal Rajbhar[3]

[1]N.R.E.C College, Department of Mathematics, Khurja- 203131 , UP , India

[2]Kirori Mal College, Department of Mathematics, University of Delhi, Delhi - 110007

[3]Atma Ram Sanatan Dharma College, Department of Mathematics, University of Delhi, Delhi – 110021

Abstract: ***The aim of this paper is to introduce a new class of normal spaces called πgβ-normal spaces, by using πgβ-open sets. We prove that πgβ-normality is a topological property and it is a hereditary property with respect to π-open, πgβ-closed subspaces. Further we obtain a characterization and preservation theorems for πgβ-normal spaces.***

Keywords: regular closed, π-closed, πgβ-closed, and β-open sets; pre β-closed, π-continuous, πgβ–continuous, π–irresolute, πgβ–irresolute and almost β–irresolute functions; πgβ-normal spaces

1. Introduction

In 1970, Levine [**7**] defined generalized closed sets in topological spaces. In 1989, Nour [**9**] introduced the notion of p-normal spaces and obtained characterization and preservation theorems for p-normal spaces. In 1990, Mahmoud and Monsef [**8**] introduced the concept of β-normal spaces. In 1995, Dontchev [**5**] introduced a new class of sets called gβ–closed sets. In 2010, Tahiliani [**10**] introduced the notion of πgβ-closed sets and its properties are studied. Recently, Thanh and Thinh [**12**] introduced the notion of πgp-normal spaces and prove that πgp-normality is a topological property and it is a hereditary property with respect to π-open, πgp-closed subspaces.

In this paper, we introduce and study a new class of normal spaces called πgβ-normal spaces by using πgβ-open sets. We prove that πgβ-normality is a topological property and it is a hereditary property with respect to π-open, πgβ-closed subspaces**.** Further we obtain a characterization and preservation theorems for πgβ-normal spaces

2. Preliminaries

Throughout this paper, spaces (X, τ), (Y, σ), and (Z, γ) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A) respectively. A subset A is said to be **regular open** [**6**] (resp. **regular closed [6]**) if A = int(cl(A)) (resp. A = cl(int(A)). The finite union of regular open sets is said to be **π-open [15]**. The complement of a π-open set is said to be **π- closed [15].** A is said to be **β-open** [**1**] if A ⊂ cl(int(cl(A))). The family of all β-open sets of X is denoted by βO(X). The complement of a β–open set is said to be **β–closed** [**1**]. The intersection of all β–closed sets containing A is called **β–closure** [2] of A, and is denoted by βcl(A). The **β-Interior** [**2**] of A, denoted by βint(A), is defined as union of all β-open sets contained in A. It is well known βcl(A) = A ∪ int(cl(int(A))) and βint(A) = A ∩ cl(int(cl(A))).

2.1 Definition

A subset A of a space X is said to be
(1) **generalized closed** (briefly **g-closed**) [**7**] if cl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(2) **generalized β- closed** (briefly **gβ-closed**) [**5**] if βcl(A) ⊂ U whenever A ⊂ U and U ∈ τ.
(3) **πgβ-closed** [**10**] if βcl(A) ⊂ U whenever A ⊂ U and U is π– open in X.
(4) **g-open** (resp**. gβ-open**, **πgβ-open**) if the complement of A is g–closed (resp. gβ-closed, πgβ–closed).

The intersection of all πgβ–closed sets containing A is called **πgβ–closure** of A, and is denoted by πgβcl(A). The **πgβ-interior** of A, denoted by πgβint(A), is defined as union of all πgβ-open sets contained in A. The family of all πgβ-open (resp. πgβ–closed) sets of X is denoted by πgβO(X) (resp. πgβC(X).

2.2 Definition

*A space X is said to be **β–normal** [**8**] (resp. **p-normal** [**9**]) if for every pair of disjoint closed subsets A, B of X, there exist disjoint β–open (resp. p-open) sets U, V of X such that A ⊂ U and B ⊂ V*

2.3 Definition

A space X is said to be π**β–normal** [**14**] (resp. **πp-normal** [**11**]) if for every pair of disjoint closed subsets A, B of X, one of which is π–closed, there exist disjoint β–open (resp. p-open) sets U, V of X such that A ⊂ U and B ⊂ V.

2.4 Definition

A subset A of a space X is said to be a β–neighborhood [8] of x if there exists a β–open set U such that x ∈ U ⊂ A.

2.5 Definition. A function f : X → Y is said to be

(a) **regular open** [**13**] if f(U) is regular open in Y for every open set U in X.

(b) **π-continuous** [**4**] if $f^{-1}(F)$ is π-closed in X for each closed set F in Y.
(c) **pre-β-closed** [**8**] (resp. **pre β–open** [**8**]) f(F) is β-closed (resp. β-open) set in for every β–closed (resp. β–open) set F in X.
(d) **πgβ-continuous** [**10**] if $f^{-1}(F)$ is πgβ-closed in X for every closed set F in Y.
(e) **πgβ-irresolute [10]** if $f^{-1}(F)$ is πgβ-closed in X for every πgβ-closed set F in Y.
(f) **almost β–irresolute** [**8**] if for each x ∈ X and β–neighborhood V of f(x) in Y, $\beta cl(f^{-1}(V))$ is neighborhood of x in X.

3. πGβ-Normal Spaces

In this section, we introduce the notion of πgβ–normal space and study some property of it. First, we begin with the following definitions and examples.

3.1 Definition

A space X is said to be **πgβ–normal** (resp. **πgp–normal** [**12**]) if for every pair of disjoint πgβ–closed (resp. πgp-closed) subsets H and K of X, there exist disjoint β–open (resp. p-open) sets U, V of X such that H ⊂ U and K ⊂ V.

Clearly, from above definitions, we have the following diagram:

πgp –normality ⇒ p-normality ⇒ πp-normality

⇓ ⇓ ⇓

πgβ–normality ⇒ β-normality ⇒ πβ-normality

Where none of the above implications is reversible as can be seen from the following examples:

3.2 Example

We consider the topology τ = {∅, {b, d}, {a, b, d}, {b, c, d}, X} on the set X = {a, b, c, d}. Then, the space X is p-normal as well as β–normal. But it is neither πgp–normal nor πgβ–normal.

3.3 Example

Let X = {a, b, c, d} and τ = {∅, {b}, {d}, {b, d}, {a, b, d}, {b, c, d}, X}. Then the space X is β–normal as well as πβ–normal but it is not p-normal.

3.4 Example

Let X = {a, b, c, d} and τ = {∅, {a}, {c}, {a, c}, {a, b, d}, {b, c, d}, X}. The pair of disjoint closed subsets of X are A = {a} and B = {c}. Also U = {a, b} and V = {c, d} are β–open sets such that A ⊂ U and B ⊂ V. Hence X is β–normal as well as πβ-normal.

3.5 Example

Let X = {a, b, c, d} and τ = {∅, {a}, {b, d}, {a, b, d}, {b, c, d}, X}. Then, the space X is β–normal.

3.6 Theorem

For a topological space X, the following are equivalent:
(a) X is πgβ-normal.
(b) For every pair of disjoint πgβ-open subsets U and V of X whose union is X, there exist β-closed subsets G and H of X such that G ⊂ U, H ⊂ V and G ∪ H = X.
(c) For every πgβ-closed set A and every πgβ-open set B in X such that A ⊂ B, there exists a β-open subset V of X such that A ⊂ V ⊂ βcl(V) ⊂ B.
(d) For every pair of disjoint πgβ-closed subsets A and B of X, there exists a β-open subset V of X such that A ⊂ V and βcl(V) ∩ B = ∅.
(e) For every pair of disjoint πgβ-closed subsets A and B of X, there exist β-open subsets U and V of X such that A ⊂ U, B ⊂ V and βcl(U) ∩ βcl(V) = ∅.

Proof

(a) ⇒ (b), (b) ⇒ (c), (c) ⇒ (d), (d) ⇒ (e) and (e) ⇒ (a).
(a) ⇒ (b) Let U and V be any πgβ–open subsets of a πgβ–normal space X such that U ∪ V = X. Then, X \ U and X \ V are disjoint πgβ–closed subsets of X. By πgβ–normality of X, there exist disjoint β–open subsets U_1 and V_1 of X such that X \ U ⊂ U_1 and X \ V ⊂ V_1. Let G = X \ U_1 and H = X \ V_1. Then, G and H are β–closed subsets in X such that G ∪ H = X.
(b) ⇒ (c). Let A be a πgβ–closed and B is πgβ–open subsets of X such that A ⊂ B. Then, A ∩ (X \ B) = ∅. Thus, X \ A and B are πgβ–open subsets of X such that (X \ A) ∪ B = X. By the Part (b), there exist β–closed subsets G and H of X such that G ⊂ (X \ A), H ⊂ B and G ∪ H = X. Thus, we obtain that A ⊂ (X \ G) ⊂ H ⊂ B. Let V = X \ G. Then V is β–open subset of X and βcl(V) ⊂ H as H is β–closed set in X. Therefore, A ⊂ V ⊂ βcl(V) ⊂ B.
(c) ⇒ (d). Let A and B be disjoint πgβ-closed subset of X. Then A⊂ X \ B, where X \ B is πgβ-open. By the part (c), there exists a β-open subset U of X such that A ⊂ U ⊂ βc1(U) ⊂ X \ B. Thus, βc1(U) ∩ B = ∅.
(d) ⇒ (e). Let A and B be any disjoint πgβ-closed subset of X. Then by the part (d), there exists a β-open set U containing A such that βcl(U) ∩ B = ∅. Since βcl(U) is β-closed, then it is πgβ-closed. Thus βcl(U) and B are disjoint πgβ-closed subsets of X. Again by the part (d), there exists a β-open set V in X such that B ⊂ V and βcl(U) ∩ βcl(V) = ∅.
(e) ⇒ (a). Let A and B be any disjoint πgβ-closed subsets of X. Then by the part (e), there exist β-open sets U and V such that A ⊂ U, B ⊂ V and βcl(U) ∩ βcl(V) = ∅. Therefore, we obtain that U ∩ V = ∅ and hence X is πgβ-normal.

3.7 Lemma

(a) The image of β-open subset under an open continuous function is β-open.
(b) The inverse image of β-open (resp. β-closed) subset under an open continuous function is β-open (resp. β-closed) subset.

3.8 Lemma [12]

The image of a regular open subset under open and closed continuous function is regular open subset.

3.9 Proposition [12]

The image of a π-open subset under open and closed continuous function is π-open set.

3.10 Proposition

If $f : X \rightarrow Y$ be an open and closed continuous bijection function and A be a πgβ-closed set in Y, then $f^{-1}(A)$ is πgβ–closed set in X.

Proof. Let A be a πgβ–closed subset of Y and U be any π–open subset of X such that $f^{-1}(A) \subset U$. Then by the **Proposition 3.9**, we have f (U) is π–open subset of Y such that $A \subset f(U)$. Since A is πgβ-closed subset of Y and f(U) is π–open set in Y, thus $\beta cl(A) \subset U$. By the **Lemma 3.7**, we obtain that $f^{-1}(A) \subset f^{-1}(\beta cl(A)) \subset U$, where $f^{-1}(\beta cl(A))$ is β-closed in X. This implies that $\beta cl(f^{-1}(A)) \subset U$. Therefore, $f^{-1}(A)$ is πgβ-closed set in X.

3.11 Theorem. πgβ-normality is a topological property.
Proof. Let X be a πgβ-normal space and $f : X \rightarrow Y$ be an open and closed bijection continuous function. We need to show that Y is πgβ-normal. Let A and B be any disjoint πgβ-closed subsets of Y. Then by the **Proposition 3.10**, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint of πgβ-closed subsets of X. By πgβ-normality of X, there exist β-open subsets U and V of X such that $f^{-1}(A) \subset U$, $f^{-1}(B) \subset V$ and $U \cap V = \varnothing$. By assumption, we have $A \subset f(U)$, $B \subset f(V)$ and $f(U) \cap f(V) = \varnothing$. By the **Lemma 3.7**, f(U) and f(V) are disjoint β-open subsets of Y such that $A \subset f(U)$ and $B \subset f(V)$. Hence, Y is πgβ-normal.

4. πGβ-normality in subspaces

4.1 Lemma. If M be an open subspace of a space X and $A \subset M$, then $\beta cl_M(A) = \beta cl_X(A) \cap M$.

4.2 Lemma [12].If M be an open subspace of a space X and $A \subset M$, then $int_M(cl_M(A) = int_X(cl_X(A) \cap M$.

4.3 Lemma [12]. If M be a π-open subspace of a space X and U be a π–open subset of X, then $U \cap M$ is π–open set in M.

4.4 Lemma. If A is both π–open and πgβ-closed subset of a space X, then A is β-closed set in X.

Proof. Since A is πgβ–closed and π–open subset of X and since $A \subset A$, then $\beta cl(A) \subset A$. But $A \subset \beta cl(A)$. Thus, $A = \beta cl(A)$. Hence, A is β–closed set in X.

4.5 Corollary. If A is both π–open and πgβ-closed subset of a space X, then A is regular closed set in X.

4.6 Theorem. Let M be a π-open subspace of a space X and $A \subset M$. If M is πgβ-closed set in X and A is πgβ-closed set in M. Then A is πgβ-closed set in X.
Proof. Suppose that M is πgβ-closed set in X and A is πgβ-closed set in M. Let U be any π–open set in X such that $A \subset U$. Then by **Lemma 4.3,** we have $A \subset M \cap U$, where $M \cap U$ is π–open set in M. Since A is πgβ–closed in M, thus $\beta cl_M(A) \subset M \cap U$. The by the **Lemma 4.1**, $\beta cl_X(A) \cap M \subset M \cap U$. By the **Lemma 4.4**, we obtain that $\beta cl_X(M) = M$. Thus, $\beta cl_X(A) \subset \beta cl_X(M) = M$. So, $\beta cl_X(A) \cap M = \beta cl_X(A)$. Hence, $\beta cl_X(M) \subset U \cap M$. Thus, $\beta cl_X(A) \subset U$. Therefore, A is πgβ–closed set in X.

4.7 Lemma. Let M be a closed domain subspace of a space X. If U is β–open set in X, then $U \cap M$ is β–open set in M.

4.8 Theorem. A πgβ–closed and π–open subspace of a πgβ–normal space is πgβ–normal.
Proof. Let M be a πgβ–closed and π–open subspace of a πgβ–normal space X. Let A and B be any disjoint πgβ–closed subsets of M. Then by **Theorem 4.6**, we have A and B are disjoint πgβ–closed sets in X. By πgβ–normality of X, there exist β–open subsets U and V of X such that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$. By the **Corollary 4.5** and **Lemma 4.7**, we obtain that $U \cap M$ and $V \cap M$ are disjoint β–open sets in M such that $A \subset U \cap M$ and $B \subset V \cap M$. Hence, M is πgβ–normal subspace of πgβ–normal space X.

5. Preservation theorems for πGβ-Normality

5.1 Definition

A function $f : X \rightarrow Y$ is said to be **π-irresolute** [3] if $f^{-1}(F)$ is π-closed in X for every π-closed set F in Y.

5.2 Theorem

If $f : X \rightarrow Y$ is π-irresolute, pre β-closed and A is a πgβ-closed subset of X, then f(A) is πgβ-closed subset of Y.

Proof. Let A be a πgβ-closed subset of X and U be any π-open set of Y such that $f(A) \subset U$. Then, $A \subset f^{-1}(U)$. Since f is π-irresolute function, then $f^{-1}(U)$ is π-open in X. Since A is πgβ-closed set in X and $A \subset f^{-1}(U)$, then $\beta cl_X(A) \subset f^{-1}(U)$. This implies that $f(\beta cl_X(A)) \subset U$. Since f is pre β-closed and $\beta cl_X(A)$ is β–closed set in X, then $f(\beta cl_X(A))$ is β–closed in Y. Thus, we have $\beta cl_Y(f(A)) \subset U$. Hence, f(A) is πgβ–closed subset of Y.

5.3 Corollary

If $f : X \rightarrow Y$ is π-continuous, pre β-closed and A is a πgβ-closed subset of X, then f(A) is πgβ-closed subset of Y.

5.4 Theorem

If $f : X \rightarrow Y$ is π-irresolute, pre β-closed and β-irresolute injection function from a space X to a πgβ–normal Y, then X is πgβ-normal.

Proof. Let A and B be any two disjoint πgβ-closed subsets of X. By the **Theorem 5.2** f(A) and f(B) are disjoint πgβ-closed subsets of Y. By πgβ–normality of Y, there exist disjoint β-open subsets U and V of Y such that $f(A) \subset U$, $f(B) \subset V$ and $U \cap V = \varnothing$. Since f is β-irresolute injection function, then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint β-open sets in X such that $A \subset f^{-1}(U)$ and $B \subset f^{-1}(V)$. Hence X is πgβ-normal.

5.5 Corollary

If $f : X \rightarrow Y$ is π-continuous, pre β-closed and β-irresolute injection function from a space X to a πgβ–normal Y, then X is πgβ-normal.

5.6 Lemma

If the bijection function $f : X \rightarrow Y$ is β–continuous and regular open, then f is πgβ–irresolute.

5.7 Theorem

If $f : X \rightarrow Y$ is πgβ-irresolute, pre β-closed bijection function from a πgβ–normal space X to a space Y, then Y is πgβ–normal.

Proof. Let A and B be any two disjoint πgβ-closed subsets of Y. Since f is πgβ–irresolute, we have $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint πgβ-closed subsets of X. By πgβ–normality of X, there exist disjoint β-open sets U and V in X such that $f^{-1}(A) \subset U$, $f^{-1}(B) \subset V$ and $U \cap V = \varnothing$. Since f is pre β–open and bijection function, we have f(U) and f(V) are disjoint β-open sets in Y such that $A \subset f(U)$, $B \subset f(V)$ and $f(U) \cap f(V) = \varnothing$. Therefore, X is πgβ-normal.

5.8 Corollary

If $f : X \rightarrow Y$ is β-continuous, regular open and pre β-open bijection function from a πgβ–normal space X to a space Y, then Y is πgβ–normal.

5.9 Theorem

If $f : X \rightarrow Y$ is a pre β–open, πgβ–irresolute and almost β–irresolute surjection function from a πgβ–normal space X onto a space Y, then Y is πgβ–normal.

Proof. Let A be a πgβ–closed subset of Y and B be a πgβ–open subset of Y such that $A \subset B$. Since f is πgβ-irresolute, we obtain that $f^{-1}(A)$ is πgβ–closed in X and $f^{-1}(B)$ is πgβ–open in X such that $f^{-1}(A) \subset f^{-1}(B)$. Since X is πgβ–normal, then by the Part (c) of the **Theorem 3.6**, there exists a β–open set U of X such that $f^{-1}(A) \subset U \subset \beta cl_X(U) \subset f^{-1}(B)$. Then, $f(f^{-1}(A)) \subset f(U) \subset f(\beta cl_X(f(U)) \subset f(f^{-1}(B)$. Since f is pre β–open, almost β–irresolute surjection, we obtain that $A \subset f(U) \subset \beta cl_Y(f(U)) \subset B$ and f(U) is β–open set in Y. Hence by the **Theorem 3.6**, we have Y is πgβ–normal.

References

[1] M. E. Abd EI-Monsef, S.N. EL Deeb and R. A. Mohamoud, β- open sets and β-continuous mappings, Bull. Fac. Assiut Univ. Sci., **12**(1983), 77-90.
[2] M. E. Abd EL-Monsef , R. A. Mahmoud, and E. R. Lashin, β- closure and β-interior, Rep., J. Fac. Of Edu. Ain Shams Univ., **10**(1986), 235.
[3] A. Aslim, A. Caksu Guler and T. Noiri, On πgs-closed sets in topological spaces, Acta Math. Hungar., **112**(2006), 275-283.
[4] J. Dontchev and T. Noiri, Quasi-normal spaces and πg-closed sets, Acta Math. Hungar., **89**(3)(2000), 211-219.
[5] J. Dontchev, On generalizing semi- preopen sets, Mem. Fac. Sci. Kochi Univ. Math., **16** (1995), 35.
[6] C. Kuratowski, Topology I, 4th, ed, In French, Hafner, New York, 1958.
[7] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo (2),**19**(1970), 89-96.
[8] R. A. Mahmoud, and M. E. Abd EI-Monsef, β-irresolute and β-topological invariant, Proc. Pakistan Acad. Sci., **27**(1990), 285-296.
[9] T. M. Nour, Contribution to the Theory of Bitopological Spaces, Ph. D. Thesis, Delhi Univ., 1989.
[10] S. Tahiliani, On πgβ-closed sets in topological spaces, Node M. 30(1) (2010), 49-55.
[11] S. A. S. Thabit and H. Kamaruihaili, πp-normality on topological spaces, Int. J. Math. Anal., 6(21) (2012), 1023-1033.
[12] L. N. Thanh and B. Q. Thinh, πgp-normal topological spaces, Journal of Advanced Studied in Topology, Vol. 4, no. 1 (2013), 48-54.
[13] J. H. Park and J. K. Park. On πgp-continuous functions in topological spaces. Chaos Solutions and Fractals, 20(2004), 467-477.
[14] M. C. Sharma and Hamant Kumar, πβ-normal Spaces, Acta Ciencia Indica, Vol. XXXVI M.no.4, (2010), 611-616.
[15] V. Zaitsev, On certain classes of topological spaces and their biocompactifications, Dokl. Akad. Nauk SSSR, **178**(1968), 778-779

Author Profile

Umesh Chand received the M.Phil (Some new mappings in Topological spaces) from Chaudhary Charan Singh University, Meerut, UP in 2011