Common Fixed Point Theorems Weak Compatible in Cone Metric Spaces

Raghu Nandan Patel¹, Damyanti Patel²

¹Department of Mathematics, Government Naveen College, Balrampur (C.G.), India
²Department of Mathematics, Government Engineering College, Bilaspur (C.G.), India

Abstract: In this paper we established common fixed point theorems for weakly compatible mappings in cone metric spaces.

AMS Subject Classification: 47H10, 54H25
Keywords: fixed point, cone metric spaces and weakly compatible mappings.

1. Introduction

The Banach contraction mapping principle is widely recognized as the source of metric fixed point theory. This contraction principle has further several generalizations in metric spaces as well as in cone metric spaces. Huang and Zhang [1] introduced the concept of cone metric space, where every pair of elements is assigned to an element of a Banach space and defined a partial order on the Banach space with the help of a subset of the Banach space called cone which satisfy certain properties.

2. Preliminary Notes

First, we recall some standard definitions and other results that will be needed in the sequel.

Definition 2.1. Let E be a real Banach space and P be a subset of E. P is called a cone if
(1) P is closed, nonempty and P ≠ ∅;
(2) a, b ∈ R, a, b ≥ 0, x, y ∈ P ⇒ ax + by ∈ P;
(3) x ∈ P and -x ∈ P ⇒ x = 0.

Let X be a nonempty set and E be a real Banach space. E with "≤" is partial ordering with respect to P. Given a cone P ⊆ E, we define a partial ordering "≤" in E by x ≤ y if x – y ∈ P. We assume cone is solid i.e. that P contains an interior of P. We assume cone is solid i.e. that P contains an interior of P. We assume cone is solid i.e. that P contains an interior of P. We assume cone is solid i.e. that P contains an interior of P. We assume cone is solid i.e. that P contains an interior of P.

Proposition 2.2 [7]: Let P be a cone in a real Banach space E.
(1) If a ∈ P and a ≤ ka, for some k ∈ [0, 1) then a = 0.
(2) If a ∈ P and a ≤ c, for all c ∈ P then a = 0.

A cone P is called normal if there is constant K > 0 such that, for all x, y ∈ E, 0 ≤ x ≤ y ⇒ ||x|| ≤ K||y||. The least value of constant K satisfying this inequality is called the normal constant of P.

Definition 2.3 [1]: Let X be a nonempty set and E be a real Banach space. Suppose that the mapping d: X × X → E satisfies
(1) 0 ≤ d(x, y), for all x, y ∈ X and d(x, y) = 0 if and only if x = y;
(2) d(x, y) = d(y, x), for all x, y ∈ X;
(3) d(x, y) ≤ d(x, z) + d(z, y), for all x, y, z ∈ X.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

For examples of cone metric spaces we refer [1, 4]. Henceforth unless otherwise indicated, P is a normal cone in real Banach space E and "≤" is partial ordering with respect to P.

Definition 2.4 [1]: Let (X, d) be a cone metric space. Let {x_n} be a sequence in X and x ∈ X.
(a) If for every c ∈ E with 0 ≤ c (or equivalently c ∈ P) there is positive integer n_0 such that for all n ≥ n_0, d(x, x_n) ≤ c there the sequence {x_n} converges to x. We denote this by x_n → x, as n → ∞ or lim_{n→∞} x_n = x.
(b) If for every c ∈ E with 0 ≤ c there is positive integer n_0 such that for all n, m > n_0, d(x, x_m) ≤ c then the sequence {x_n} is called a Cauchy sequence in X.

(X, d) is called a complete cone metric space, if every Cauchy sequence in X is convergent in X.

Lemma 2.5 [1]: Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let {x_n} be a sequence in X, then {x_n} converges to x if and only if d(x, x_n) → 0 (n → ∞).

Lemma 2.6 [1]: Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let {x_n} be a sequence in X, then {x_n} is a Cauchy sequence if and only if d(x, x_m) → 0 (n, m → ∞).

Definition 2.7: A function f: P → P is called subadditive if for all x, y ∈ P,

f(x + y) ≤ f(x) + f(y).

Definition 2.8: If Y be any partially ordered set with relation "≤" and Ψ : Y → Y, we say that Ψ is non decreasing if x, y ∈ Y, x ≤ y ⇒ Ψ(x) ≤ Ψ(y).

Definition 2.9 [2]: Let Ψ : P → P be a vector valued function then Ψ is called MS-Altering function if
(a) Ψ is non decreasing, subadditive, continuous and sequentially convergent;
(b) Ψ(a) = 0 if and only if a = 0.
We replace conditions (a) and (b) by weaker conditions and define cone altering function as follows

**Definition 2.10:** Let $\Psi : P \to P$ be a vector valued function then $\Psi$ is called cone altering function if
(a) $\Psi$ is non decreasing, subadditive;
(b) $\Psi(a_n) \to 0$ if and only if $a_n \to 0$, for any sequence $\{a_n\}$ in $P$.

Note that for cone altering function $\Psi$ on normal cone $P$, $\Psi(a) = 0$ if and only if $a = 0$.

**Definition 2.11:** Let $X$ be any nonempty set, $f, g : X \to X$ be mappings. A point $w \in X$ is called point of coincidence of $f$ and $g$ if there is $x \in X$ such that $fx = gx = w$.

**Definition 2.12:** Let $X$ be any nonempty set, $f, g : X \to X$ be mappings. Pair $(f, g)$ is called weakly compatible if $x \in X$, $fx = gx = w$ and a unique point $z$ such that $Aw = Sw = w$ and a unique point $z$.

**Lemma 2.13:** Let $(X, d)$ be a cone metric space with a normal cone $P$ in a real Banach space $E$, $\Psi$ is a cone altering function and $k_1, k_2, k > 0$. If $x_n \to x$, $y_n \to y$ in $X$ and $ka \leq k_1 \Psi[d(x_n, x)] + k_2 \Psi[d(y_n, y)]$, then $a = 0$.

**Lemma 2.14:** Let $(X, d)$ be a cone metric space with $\Psi : P \to P$ be a cone altering function such that

**Theorem 3.1:** Let $(X, d)$ be a cone metric space and $P$ be a normal cone in a real Banach space $E$, $\Psi$ is a cone altering function and $k_1, k_2, k > 0$. If $x_n \to x$, $y_n \to y$ in $X$ and $ka \leq k_1 \Psi[d(x_n, x)] + k_2 \Psi[d(y_n, y)]$, then $a = 0$.

**Main Results**

**Theorem 3.2:** Let $(X, d)$ be a cone metric space with normal cone $P$ and let $A, B, S$ and $T$ be self mappings of $X$, $\Psi : P \to P$ be cone altering function such that $\Psi[d(w, z)] = \Psi[d(Aw, Bz)]$.

Assume that $w \neq z$. We have $\Psi[d(w, z)] = \Psi[d(Aw, Bz)]$.

**Proof:** Let the pairs $(A, S)$ and $(B, T)$ be weakly compatible and have a unique point of coincidence then it is unique common fixed point of $A$ and $S$.

**Theorem 3.3:** The pairs $(A, S)$ and $(B, T)$ be weakly compatible and have a unique point of coincidence then it is unique common fixed point of $A$ and $S$. Since $(a_1 + a_2 + a_3) < 1$ hence by proposition 2.2, we have $\Psi[d(Ax, Ay)] = 0 i.e. d(Ax, Ay) = 0$ or $Ax = Ay$.

Therefore $Ax = Sx = Ay = Ty$. Suppose that there is another point $z$ such that $Az = Sz$ then by (3.1.3) we have $Az = Sz = Ay = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of $A$ and $S$. By Lemma 2.14 $w$ is the only common fixed point of $A$ and $S$. Similarly there is a unique point $z$ such that $z = Az = Tz$.

Assume that $w \neq z$. We have $\Psi[d(w, z)] = \Psi[d(Aw, Bz)]$.

**Proof:** Let the pairs $(A, S)$ and $(B, T)$ be weakly compatible and have a unique point of coincidence then it is unique common fixed point of $A$ and $S$. Since $(a_1 + a_2 + a_3) < 1$ hence by proposition 2.2, we have $\Psi[d(Ax, Ay)] = 0 i.e. d(Ax, Ay) = 0$ or $Ax = Ay$.

Therefore $Ax = Sx = Ay = Ty$. Suppose that there is another point $z$ such that $Az = Sz$ then by (3.2.3) we have $Ax = Sz = Ay = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of $A$ and $S$. By Lemma 2.14 $w$ is the only common fixed point of $A$ and $S$. Similarly there is a unique point $z$ such that $z = Az = Tz$.

Assume that $w \neq z$. We have $\Psi[d(w, z)] = \Psi[d(Aw, Bz)]$.
+a_5\psi[d(z, w)]
\quad= (a_1 + a_4 + a_5) \psi[d(z, w)]
since (a_1 + a_4 + a_5) < 1 hence by proposition 2.2, we have
\psi[d(w, z)] = 0 i.e. d(w, z) = 0 or w = z by Lemma 2.14 and
z is a unique common fixed point of A, B, S and T.

References

[1] L.G. Huang, X.Zhang, Cone Metric Spaces and Fixed
Point Theorems in Cone Metric Space by Altering
Distances; International Mathematical Forum, 54(6)
Point Theorems in Cone Metric Space by Altering
[4] Sh. Rezapour, R. Hambarbari, Some notes on the paper
Cone Metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 345(2008),
719 – 724.
[5] A.K Dubey and A. Narayan, Cone Metric Spaces and
Fixed Point Theorems for Pair of contractive maps,
the Paper “Cone Metric Spaces and Fixed Point
Theorems of Contractive Mappings”, Int. J. of Applied
Mathematical Research, 2(1), (2013), 84-90.
[7] D. Ilic, V Rakcevic, Qusai-contraction on a cone metric
space, applied Mathematics Letter, 22(5) (2009), 728 –
731.

Author Profile

Dr Raghu Nandan Patel is Assistant Professor,
Department of Mathematics, Govt, Naveen College
Balrampur (C. G.) India

Dr Damyanti Patel is Lecture in Department of
Mathematics, Government Engineering College,
Bilaspur (C. G.) India