Zero-Free Region for Polynomials with Restricted Coefficients

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Abstract: In this paper we prove some extension of the Eneström-Kakeya theorem says that. Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that \( 0 < a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \) then all the zeros of \( P(z) \) lie in \( |z| \leq 1 \). By relaxing the hypothesis of this result in several ways and obtain zero-free regions for polynomials with restricted coefficients and there by present some interesting generalizations and extensions of the Enestrom-Kakeya Theorem.

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1. Introduction

The well known Results Eneström-Kakeya theorem [1, 2] in theory of the distribution of zeros of polynomials is the following.

**Theorem (A₁).** Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that \( 0 < a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \) then all the zeros of \( P(z) \) lie in \( |z| \leq 1 \).

Applying the above result to the polynomial \( z^n P(z) \) we get the following result:

**Theorem (A₂).** If \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \) such that \( 0 < a_0 \leq a_{n-1} \leq a_{n-2} \leq \ldots \leq a_0 \) then \( P(z) \) does not vanish in \( |z| < 1 \).

In the literature [3-10], there exist several extensions and generalizations of the Eneström-Kakeya Theorem.

In this paper we give generalizations of the above mentioned results. In fact we prove the following results:

**Theorem 1.** Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m < n \) with real coefficients such that

\[
\begin{align*}
a_0 &\geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \ldots \geq a_{n-m-1} \geq a_{n-m} \\
&\geq a_{n-m+1} \geq \ldots \geq a_{n-2} \geq a_{n-1} \geq a_n
\end{align*}
\]

if both \( n \) and \( (n-m) \) are even or odd

(OR)

\[
\begin{align*}
a_0 &\geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \ldots \geq a_{n-m-1} \geq a_{n-m} \\
&\geq a_{n-m+1} \geq \ldots \geq a_{n-2} \geq a_{n-1} \geq a_n
\end{align*}
\]

if \( n \) is even and \( (n-m) \) is odd (or) if \( n \) is odd and \( (n-m) \) is even

then (i) all the zeros of \( P(z) \) does not vanish in the disk

\[
|z| < \frac{a_0}{a_0 + |a_n| - a_n + S_1}
\]

if both \( n \) and \( (n-m) \) are even or odd

where

\[
S_1 = 2 \left[ \left( a_2 + a_4 + \ldots + a_{n-m-2} + a_{n-m} \right) - (a_1 + a_3 + \ldots + a_{n-m-3} + a_{n-m-1}) \right]
\]

(ii) all the zeros of \( P(z) \) does not vanish in the disk

\[
|z| < \frac{a_0}{a_0 + |a_n| - a_n + S_2}
\]

if \( n \) is even and \( (n-m) \) is odd (or) if \( n \) is odd and \( (n-m) \) is even

where

\[
S_2 = 2 \left[ \left( a_2 + a_4 + \ldots + a_{n-m-3} + a_{n-m-1} \right) - (a_1 + a_3 + \ldots + a_{n-m-4} + a_{n-m-2}) \right]
\]

**Corollary 1.** Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m < n \) with positive real coefficients such that

\[
\begin{align*}
a_0 &\geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \ldots \geq a_{n-m-1} \geq a_{n-m} \\
&\geq a_{n-m+1} \geq \ldots \geq a_{n-2} \geq a_{n-1} \geq a_n
\end{align*}
\]

if both \( n \) and \( (n-m) \) are even or odd

(OR)

\[
\begin{align*}
a_0 &\geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \ldots \geq a_{n-m-1} \geq a_{n-m} \\
&\geq a_{n-m+1} \geq \ldots \geq a_{n-2} \geq a_{n-1} \geq a_n
\end{align*}
\]

if \( n \) is even and \( (n-m) \) is odd (or) if \( n \) is odd and \( (n-m) \) is even

then (i) all the zeros of \( P(z) \) does not vanish in the disk

\[
|z| < \frac{a_0}{a_0 + S_1}
\]

if both \( n \) and \( (n-m) \) are even or odd

where

\[
S_1 = 2 \left[ \left( a_2 + a_4 + \ldots + a_{n-m-2} + a_{n-m} \right) - (a_1 + a_3 + \ldots + a_{n-m-3} + a_{n-m-1}) \right]
\]

(ii) all the zeros of \( P(z) \) does not vanish in the disk

\[
|z| < \frac{a_0}{a_0 + S_2}
\]
Theorem 2. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$ and $0 \leq m < n$ with real coefficients such that

$$a_0 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n$$

if both $n$ and $(n-m)$ are even or odd (OR)

$$a_0 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n$$

if $n$ is even and $(n-m)$ is odd (OR) if $n$ is odd and $(n-m)$ is even

then (i) all the zeros of $P(z)$ does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n|-a_0+a_n+T_1}$$

if both $n$ and $(n-m)$ are even or odd

where

$$T_1 = 2\left( (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) \right)$$

(ii) all the zeros of $P(z)$ does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n|-a_0+a_n+T_2}$$

if $n$ is even and $(n-m)$ is odd (OR) if $n$ is odd and $(n-m)$ is even

where

$$T_2 = 2\left( (a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) \right)$$

Corollary 2. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$ and $0 \leq m < n$ with positive real coefficients such that

$$a_0 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n$$

if both $n$ and $(n-m)$ are even or odd (OR)

$$a_0 \leq a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n$$

if $n$ is even and $(n-m)$ is odd (OR) if $n$ is odd and $(n-m)$ is even

then (i) all the zeros of $P(z)$ does not vanish in the disk

$$|z| < \frac{a_0}{a_0+T_1-a_0}$$

if both $n$ and $(n-m)$ are even or odd

where

$$T_1 = 2\left( (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) \right)$$

(ii) all the zeros of $P(z)$ does not vanish in the disk

$$|z| < \frac{a_0}{T_2-a_0}$$

if both $n$ and $(n-m)$ are even or odd

where

$$T_2 = 2\left( (a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) \right)$$

Remark 1. By taking $a_i > 0$ for $i = 0, 1, 2, \ldots, n$, in theorem 1, then it reduces to Corollary 1.

Theorem 3. Let let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$ and $0 \leq m < n$ with real coefficients such that

$$a_0 \geq a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_{n-m-1} \geq a_{n-m}$$

if both $n$ and $(n-m)$ are even or odd (OR)

$$a_0 \geq a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_{n-m-1} \leq a_{n-m}$$

if $n$ is even and $(n-m)$ is odd (OR) if $n$ is odd and $(n-m)$ is even

then (i) all the zeros of $P(z)$ does not vanish in the disk

$$|z| < \frac{a_0}{a_0+T_1-a_0}$$

if both $n$ and $(n-m)$ are even or odd

where

$$T_1 = 2\left( (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) \right)$$

(ii) all the zeros of $P(z)$ does not vanish in the disk

$$|z| < \frac{a_0}{a_0+T_2-a_0}$$

if both $n$ and $(n-m)$ are even or odd

where

$$T_2 = 2\left( (a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) \right)$$

Remark 2. By taking $a_i > 0$ for $i = 0, 1, 2, \ldots, n$, in theorem 2, then it reduces to Corollary 4.
if \( n \) is even and \((n-m)\) is odd (or) if \( n \) is odd and \((n-m)\) is even

where \( U_2 = 2\left( a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}\right) - \left(a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m}\right) \).

**Remark 3.** By taking \( a_i > 0 \) for \( i = 0, 1, 2, ..., n \) in theorem 3, then it reduces to Corollary 3.

**Theorem 4.** Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m \leq n \) with real coefficients such that

\[
a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n
\]

if both \( n \) and \((n-m)\) are even or odd (OR)

\[
a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n
\]

if \( n \) is even and \((n-m)\) is odd (or) if \( n \) is odd and \((n-m)\) is even

\[
|V_1| = 2\left( a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}\right) - \left(a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}\right)
\]

(ii) ) all the zeros of \( P(z) \) does not vanish in the disk \(|z| < \frac{|a_0| + |a_{n-m} + a_{n-m+1}|}{2a_n + a_{n-m+1} + V_1}\)

if both \( n \) and \((n-m)\) are even or odd

\[
|V_1| = 2\left( a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}\right) - \left(a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}\right)
\]

**Corollary 4.** Let \( P(z) = \sum_{i=0}^{n} a_i z^i \) be a polynomial of degree \( n \geq 2 \) and \( 0 \leq m \leq n \) with positive real coefficients such that

\[
a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n
\]

if both \( n \) and \((n-m)\) are even or odd (OR)

\[
a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n
\]

if \( n \) is even and \((n-m)\) is odd (or) if \( n \) is odd and \((n-m)\) is even

then (i) all the zeros of \( P(z) \) does not vanish in the disk \(|z| < \frac{|a_0|}{2a_n + a_{n-m+1} + V_1}\)

if both \( n \) and \((n-m)\) are even or odd

\[
|V_1| = 2\left( a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}\right) - \left(a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}\right)
\]

(ii) ) all the zeros of \( P(z) \) does not vanish in the disk \(|z| < \frac{|a_0|}{2a_n + a_{n-m+1} + V_1}\)

if both \( n \) and \((n-m)\) are even or odd

\[
V_2 = 2\left( a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-2}\right) - \left(a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m-1}\right).
\]

**Remark 4.** By taking \( a_i > 0 \) for \( i = 0, 1, 2, ..., n \) in theorem 4, then it reduces to Corollary 4.

2. Proofs of the Theorems

**Proof of the Theorem 1.**

Let \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \) be a polynomial of degree \( n \)

Let us consider the polynomial \( J(z) = z^n P(\frac{1}{z}) \)

and \( R(z) = (z - 1)J(z) \) so that

\[
R(z) = (z - 1)\left( a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-m} z^{n-m} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \cdots + a_{n-1} z + a_n \right)
\]

Also if \( |z| > 1 \) then \( \frac{1}{|z|^{n-i}} < \frac{1}{|z|^{n-i}} \) for \( i = 0, 1, 2, ..., n-1 \).

Now \( |R(z)| \geq |a_0||z|^{n-1} - \left( |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{n-1} - a_n||z| \right) \)

\[
\geq |a_0||z|^n \left( |z| - \frac{1}{|a_0|} ||a_0 - a_1| + \frac{|a_1 - a_2|}{|z|^{n-1}} + \frac{|a_2 - a_3|}{|z|^{n-2}} + \cdots + \frac{|a_{n-2} - a_{n-1}|}{|z|^{n-1}} + \frac{|a_{n-1} - a_n|}{|z|^2} \right)
\]

\[
\geq |a_0||z|^n \left( |z| - \frac{1}{|a_0|} \left( |a_0 - a_1| + |a_1 - a_2| + \cdots + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n| \right) \right)
\]

if both \( n \) and \((n-m)\) are even or odd

\[
= |a_0||z|^n \left( |z| - \frac{1}{|a_0|} \left( a_0 - a_1 + a_1 - a_2 + a_2 - a_3 + a_3 - a_4 + \cdots + a_{n-2} - a_{n-1} + a_{n-1} - a_n \right) \right)
\]

\[
= \frac{|a_0||z|^n}{|a_0|} \left( a_0 - a_1 + a_1 - a_2 + a_2 - a_3 + a_3 - a_4 + \cdots + a_{n-2} - a_{n-1} + a_{n-1} - a_n \right)
\]

\[
\Rightarrow R(z) > 0 \text{ if } |z| > \frac{1}{|a_0|} \left( a_0 + |a_n| - a_n + S_1 \right)
\]
This shows that all the zeros of $R(z)$ whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|} \left\{ a_0 + |a_n| - a_n + S_1 \right\}$$

But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of $R(z)$ and hence $J(z)$ lie in

$$|z| \leq \frac{1}{|a_0|} \left\{ a_0 + |a_n| - a_n + S_1 \right\}$$

Since $P(z) = z^n J\left(\frac{1}{z}\right)$ it followed by replacing $z$ by $\frac{1}{z}$, all the zeros of $P(z)$ lie in

$$|z| \geq \frac{|a_0|}{a_0 + |a_n| - a_n + S_1},$$

if both $n$ and $(n-m)$ are even or odd.

Hence all the zeros $P(z)$ does not vanish in the disk

$$|z| < \frac{|a_0|}{a_0 + |a_n| - a_n + S_2}$$

if both $n$ and $(n-m)$ are even or odd where

$$S_2 = 2\left\{ a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m} - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) \right\}$$

Similarly we can also prove for if $n$ is even and $(n-m)$ is odd (or) if $n$ is odd and $(n-m)$ is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros of $P(z)$ does not vanish in the disk.

$$|z| < \frac{|a_0|}{a_0 + |a_n| - a_n + S_2}$$

if $n$ is even and $(n-m)$ is odd (or) if $n$ is odd and $(n-m)$ is even where

$$S_2 = 2\left\{ a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1} - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2}) \right\}$$

This completes the proof of the Theorem 1.

**Proof of the Theorem 2.**

Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n$

Let us consider the polynomial $J(z) = z^n P\left(\frac{1}{z}\right)$

and $R(z) = (z-1) J(z)$ so that

$$R(z) = (z-1) \left\{ a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{m-1} + a_m z^m + a_{m+1} z^{m+1} + \cdots + a_{n-1} z + a_n \right\}$$

$$= a_0 z^{n+1} - \left\{ (a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{m-1} - a_m) z^{m-1} + (a_m - a_{m+1}) z^m + \cdots + (a_{n-1} - a_n) z + a_n \right\}$$

Also if $|z| > 1$ then $|\frac{1}{z^{i-n}}| < \frac{1}{i} \text{ for } i = 0, 1, 2, \ldots, n - 1.$

Now

$$|R(z)| \geq |a_0| |z|^{n+1} - \left\{ |a_0 - a_1| |z|^n + |a_1 - a_2| |z|^{n-1} + \cdots + |a_{m-1} - a_m| |z|^{m-1} + |a_m - a_{m+1}| |z|^m + \cdots + |a_{n-1} - a_n| |z| + |a_n| \right\}$$

$$\geq |a_0| |z|^n \left\{ |z| - \frac{1}{|a_0|} \left\{ |a_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-1} - a_n| + |a_n| \right\} \right\}$$

$$\geq |a_0| |z|^n \left\{ |z| - \frac{1}{|a_0|} \left\{ |a_0 - a_1| + |a_1 - a_2| + |a_3 - a_4| + \cdots + |a_{n-m-1} - a_{n-m}| + |a_{n-m} - an-m+1+\cdots+an-3-an-2+an-2-an-1+an-1-an+|a_n| \right\} \right\}$$

if both $n$ and $(n-m)$ are even or odd

$$= |a_0| |z|^n \left\{ |z| - \frac{1}{|a_0|} \left\{ |a_n| - a_n - a_0 + T_1 \right\} \right\}$$

where

$$T_1 = 2\left\{ a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1} - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) \right\}$$

$$\Rightarrow R(z) > 0 \text{ if } |z| > \frac{1}{|a_0|} \left\{ |a_n| - a_n - a_0 + T_1 \right\}$$

This shows that all the zeros of $R(z)$ whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|} \left\{ |a_n| - a_n - a_0 + T_1 \right\}$$

But those zeros of $R(z)$ whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of $R(z)$ and hence $J(z)$ lie in

$$|z| \leq \frac{1}{|a_0|} \left\{ |a_n| - a_n - a_0 + T_1 \right\}$$

Since $P(z) = z^n J\left(\frac{1}{z}\right)$ it followed by replacing $z$ by $\frac{1}{z}$, all the zeros of $P(z)$ lie in

$$|z| \geq \frac{|a_0|}{|a_n| - a_n - a_0 + T_1},$$

if both $n$ and $(n-m)$ are even or odd.

Hence all the zeros $P(z)$ does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| - a_n - a_0 + T_1}$$
if both \( n \) and \((n-m)\) are even or odd

where

\[
T_1 = 2 \left[ (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) \right]
\]

Similarly we can also prove for if \( n \) is even and \((n-m)\) is odd (or) if \( n \) is odd and \((n-m)\) is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros \( P(z) \) does not vanish in the disk.

\[
|z| < \frac{|a_0|}{|a_n| - a_n - a_0 + T_2}
\]

if \( n \) is even and \((n-m)\) is odd (or) if \( n \) is odd and \((n-m)\) is even

where

\[
T_2 = 2 \left[ (a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \cdots + a_{n-m+1} + a_{n-m-1}) \right]
\]

This completes the proof of the Theorem 2.

**Proof of the Theorem 3.**

Let \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \) be a polynomial of degree \( n \)

Let us consider the polynomial \( J(z) = z^n P(\frac{1}{z}) \)

and \( R(z) = (z - 1) J(z) \) so that

\[
R(z) = (z - 1) \left( a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{n-m} + a_m z^{n-m+1} + a_{m+1} z^{n-m+2} + \cdots + a_n \right)
\]

\[
= a_0 z^{n+1} - \left[ (a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{m-1} - a_m) z^{n-m} + (a_m - a_{m+1}) z^{n-m+1} + \cdots + (a_n - a_0) \right]
\]

also if \( |z| > 1 \) then \( \frac{1}{|z|^n} < f \) for \( f = 0, 1, 2, \ldots, n-1 \).

Now

\[
|R(z)| \geq |a_0| |z|^{n+1} - \left[ |a_0 - a_1| |z|^n + |a_1 - a_2| |z|^{n-1} + \cdots + |a_{m-1} - a_m| |z|^{n-m} + |a_m - a_{m+1}| |z|^{n-m+1} + \cdots + |a_n - a_0| \right]
\]

\[
\geq |a_0||z|^{n+1} \left[ |z| = 1 \frac{|a_0 - a_1|}{|a_0|} + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^{n+1}} + \cdots + \frac{|a_{m-1} - a_m|}{|z|^{n-m}} + \frac{|a_m - a_{m+1}|}{|z|^{n-m+1}} + \cdots + \frac{|a_n - a_0|}{|z|^n} \right]
\]

\[
\geq |a_0||z|^{n+1} \left[ |z| = 1 \frac{|a_0 - a_1|}{|a_0|} + \frac{|a_1 - a_2|}{|z|} + \frac{|a_2 - a_3|}{|z|^{n+1}} + \cdots + \frac{|a_{m-1} - a_m|}{|z|^{n-m}} + \cdots + \frac{|a_n - a_0|}{|z|^n} \right]
\]

if \( n \) is even and \((n-m)\) is odd (or) if \( n \) is odd and \((n-m)\) is even

\[
= |a_0||z|^{n+1} \left[ |z| - \frac{1}{|a_0|} |a_0 - a_1| + \frac{a_2 - a_3}{|z|} + \cdots + \frac{|a_{m-1} - a_m|}{|z|^{n-m}} + \cdots + \frac{|a_n - a_0|}{|z|^n} \right]
\]

\[
\geq |a_0||z|^{n+1} \left[ |z| - \frac{1}{|a_0|} |a_0 - a_1| + (a_2 - a_3) + \cdots + (a_{m-1} - a_m) + (a_m - a_{m+1}) \right]
\]

\[
\geq |a_0||z|^{n+1} \left[ |z| - \frac{1}{|a_0|} |a_0 - a_1| + (a_2 - a_3) + \cdots + (a_{m-1} - a_m) + (a_m - a_{m+1}) \right]
\]

This completes the proof of the Theorem 3.

**Proof of the Theorem 4.**

Let \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \) be a polynomial of degree \( n \)
Let us consider the polynomial \( f(z) = z^n P(z) \)

and \( R(z) = (z - 1) f(z) \) so that

\[
R(z) = (z - 1) (a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \cdots + a_{n-1} z + a_n)
\]

\[
= a_0 z^{n+1} - \{ (a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{m-1} - a_m) z^{n-m} + \cdots + (a_{n-1} - a_n) z + a_n \}
\]

Also if \( |z| > 1 \), then \( \frac{1}{|z|^{n-i}} < f \) for \( i = 0, 1, 2, \ldots, n - 1 \).

Now

\[
|R(z)| \geq |a_0| |z|^{n+1} - \{ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m} + |a_{m+1}||z|^{n-m-1} + \cdots + |a_{n-1} - a_n||z| + |a_n| \}
\]

\[
\geq |a_0| |z|^n \{ |z| - \frac{1}{|a_0|} \{ |a_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-1} - a_n| + |a_n| \}
\]

\[
\geq |a_0| |z|^n \{ |z| - \frac{1}{|a_0|} \{ |a_0| + |a_1 - a_2| + |a_3 - a_2| + \cdots + |a_{n-1} - a_m| + |a_n - a_{m+1}| + \cdots + |a_{n-1} - a_n| + |a_n| \}
\]

\[
R(z) > 0 \text{ if } |z| > \frac{1}{|a_0|} \{ |a_0| + a_n - a_0 + V_1 \}
\]

This shows that all the zeros of \( R(z) \) whose modulus is greater than 1 lie in the closed disk

\[
|z| \leq \frac{1}{|a_0|} \{ |a_0| + a_n - a_0 + V_1 \}
\]

But those zeros of \( R(z) \) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of \( R(z) \) and hence \( J(z) \) lie in

\[
|z| \leq \frac{1}{|a_0|} \{ |a_0| + a_n - a_0 + V_1 \}
\]

Since \( P(z) = z^n J(z) \) it followed by replacing \( z \) by \( \frac{1}{z} \),

all the zeros of \( P(z) \) lie in

\[
|z| \geq \frac{1}{|a_0|} \{ |a_0| + a_n - a_0 + V_1 \}
\]

if both \( n \) and \( (n-m) \) are even or odd.

Hence all the zeros \( P(z) \) does not vanish in the disk

\[
|z| < \frac{|a_0|}{|a_0| + a_n - a_0 + V_1}
\]

if both \( n \) and \( (n-m) \) are even or odd

where \( V_1 = 2 \{ (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) \}
\]

Similarly we can also prove for if \( n \) is even and \( (n-m) \) is odd (or) if \( n \) is odd and \( (n-m) \) is even degreepolynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros \( P(z) \) does not vanish in the disk

\[
|z| < \frac{|a_0|}{|a_0| + a_n - a_0 + V_2}
\]

if \( n \) is even and \( (n-m) \) is odd (or) if \( n \) is odd and \( (n-m) \) is even

where \( V_2 = 2 \{ (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) \} \)

This completes the proof of the Theorem 4.

References

[1] G.Eneström,Remarquesur un th\'eorem\'erelatif aux racines de l’equation \( a_n z^n + \cdots + a_0 = 0 \) o\'u tous les coefficient sont et positifs, Tôhoku Math.J 18 (1920),34-36.


