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On a New Subclass of Harmonic Meromorphic Functions

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Abstract: ***In the present paper , we introduce a new class of harmonic meromorphic functions*** $WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$***,we obtain some geometric properties , like , coefficient inequality , extreme point,convolution and convex combinotion for this class***

Kewords: Harmonic function , Meromorphic functions , starlike function

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1. Introduction

Let a continuous function $f = u + iv$ be a complex valued harmonic function in a complex domain $\mathbb{C}$, if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$, we can write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co- analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see Clunie and Sheil-Small(see Clunie and Sheil-Small [5]). Denote by $R_H(1)$the class of functions $f = h + \bar{g}$ that are harmonic, meromorphic orientation preserving and univalent in the $\bar{U} = \{z \in \mathbb{C}: |z| > 1\}$. So $f = h + \bar{g} \in R_H(1)$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \bar{g} \in R_H(1)$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^{-n} \ , g(z) = \sum_{n=1}^{\infty} b_n z^{-n} \ , |b_1| < 1. (1)$$

We denote by the class $WR_H(a,b;\lambda,\gamma,v,\square)$the class of all functions of the form (1) that satisfies the condition:

$$Re\left\{v + \frac{a}{b}\left[\left(\frac{(1+\square)(zf'(z) + \lambda z^2 f''(z))}{(1-\lambda)f(z) + \lambda f'(z)} - \square\right) - v\right]\right\} \geq \gamma \, , (2)$$

where $|a|, |b| \leq 1 \, , \frac{1}{2} \leq \lambda < 1 \, , 0 \leq \gamma < v < 1 \, , 0 < \square < 1$.

Also Let $WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$ be subclass of $WR_H(a,b;\lambda,\gamma;v,\square)$ consistiog$f = h + \bar{g}$ in which h and g are of the form

$$h(z) = z - \sum_{n=1}^{\infty} a_n z^{-n} \ ; g(z) = -\sum_{n=1}^{\infty} b_n z^{-n} \ , a_n \geq 0 \, , b_n \geq 0 \, . (3)$$

We obtain sufficient coefficient condition for harmonic meromorphic functions $f = h + \bar{g}$ to be in the class $WR_H(a,b;\lambda,\gamma;v,\square)$. We also show that this coefficient condition is also necessary for $f \in WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$.We also obtain distortion bounds, extreme points, convolution condition and convex combination for functions in $WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$.

2. Main Result

We begin with a sufficient condition for function in $WR_H(a,b;\lambda,\gamma;v,\square)$.

Theorem (1): Let $f = h + \bar{g}$ with h and g are given by (1) if

$$\sum_{n=2}^{\infty}\Big(\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\Big)|a_n| + \sum_{n=1}^{\infty}\Big(\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\Big)|b_n| \leq |a|(1-v) + |b|(v-\gamma), (4)$$

where $|a|, |b| \leq 1 \, , \frac{1}{2} \leq \lambda < 1 \, , 0 \leq \gamma < v < 1 \, , 0 < \square < 1$.
Then f is harmonic Univalent in $\bar{U}$ and
$f \in WR_H(a,b;\lambda,\gamma;v,\square)$.

Proof: To show that $\in WR_H(a,b;\lambda,\gamma;v,\square)$, we notice according to(2) , we must have

$$Re\left\{\frac{A(z)}{B(z)}\right\} > \gamma \ where$$

$$A(z) = vb\big((1-\lambda)f(z) + \lambda zf'(z)\big) + a(1+\square)\big(zf'(z) + \lambda z^2 f''(z)\big) - a(\square + v)(1-\lambda)f(z) + \lambda zf'(z))$$

$$B(z) = b\big((1-\lambda)f(z) + \lambda zf'(z)\big)$$

Using the fact that $Re(w) \geq \gamma$ if and only if $|1 - \gamma + w| \geq |1 + \gamma - w|$.
For $0 \leq \gamma < 1$ it is enough to show that .

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0$$

Differentiating$h$ and g and Subalituting in the above inequality we obtain

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)|$$

$$= \Bigg|(v(b-a)+a)z + \sum_{n=2}^{\infty}\Big(vb\big(1-\lambda(n+1)\big) + a(v+\square+2n)(\lambda(n+1)-1)\Big)a_n z^{-n} + \sum_{n=1}^{\infty}(vb\,(1-\lambda(n+1)) + a(v+\square+2n)(\lambda(n+1)-1)b_n(\bar{z})^{-n} + (1-\gamma)\Big(bz + \sum_{n=2}^{\infty} b\big(1-\lambda(n+1)\big)a_n\, z^{-n} + \sum_{n=1}^{\infty} b\big(1-\lambda(n+1)\big)b_n(\bar{z})^{-n}\Bigg|$$

$$- \Bigg|(v(b-a)+a)z + \sum_{n=2}^{\infty}\Big(vb\big(1-\lambda(n+1)\big) + a(v+\square+2n)(\lambda(n+1)-1)\Big)a_n z^{-n} + \sum_{n=1}^{\infty}(vb\,(1-\lambda(n+1)) + a(v+\square+2n)(\lambda(n+1)-1)b_n(\bar{z})^{-n} - (1+\gamma)\Big(bz + \sum_{n=2}^{\infty} b\big(1-\lambda(n+1)\big)a_n\, z^{-n} + \sum_{n=1}^{\infty} b\big(1-\lambda(n+1)\big)b_n(\bar{z})^{-n}\Bigg|$$

$$\geq 2\big(|a|(1-v)+|b|(v-\gamma)\big)|z| - \sum_{n=2}^{\infty}\Big(2\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + 2(\lambda(n+1)-1)(v-\gamma)|b|\Big)|a_n||z^{-n}| - \sum_{n=1}^{\infty}\Big(2\big(1-\lambda(n+1)\big)(v+m+2n)|a| + 2(\lambda(n+1)-1)(v-\gamma)|b|\Big)|b_n|\,.$$

The harmonic Univalent function .

$$f(z) = z + \sum_{n=2}^{\infty}\frac{x_n}{\big[\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\big]}(z)^{-n} + \sum_{n=1}^{\infty}\frac{\overline{y_n}}{\big[\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\big]}(\bar{z})^{-n} \quad (5)$$

Where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = |a|(1-v)+|b|(v-\gamma)$.

Show that the Coefficient bound given by (4) is sharp
The function of form (5) are in the class $WR_H(a,b;\lambda,\gamma;v,\square)$ because

$$\sum_{n=2}^{\infty}\big[\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\big]\frac{|x_n|}{\big[\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\big]} + \sum_{n=1}^{\infty}\big[\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\big]\frac{|y_n|}{\big[\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\big]}$$

$$= \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = |a|(1-v)+|b|(v-\gamma)\,.$$

The restriction placed in Theorem (1) on moduli of the Coefficients of $f = h + \bar{g}$ enables us to Conclude for arbitrary rotation of the Coefficients of f that the resulting functions would still be harmonic Univalent and $f \in WR_H(a,b;\lambda,\gamma;v,\square)$.

In the following Theorem ,it is shown that the condition (4) is also necessary for function in $WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$.

Theorem (2): Let $f = h + \bar{g}$ with h and g are given by (3) . A necessary and sufficient condition for f to be in $WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$ is that

$$\sum_{n=2}^{\infty}\Big(\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\Big)|a_n| + \sum_{n=1}^{\infty}\Big(\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\Big)|b_n| \leq |a|(1-v)+|b|(v-\gamma). \quad (6)$$

Where $|a|,|b| \leq 1$, $\frac{1}{2} \leq \lambda < 1$, $0 \leq \gamma < v < 1$, $0 < \square < 1$.

Proof: In view of Theorem 1 .We need only show that $f \notin WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$ if the coefficient inequality (6) does not hold .

We note that if $f \in WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$ we must have

$$Re\left\{v+\frac{a}{b}\left[\left(\frac{(1+\square)\left(zf'(z)+\lambda z^2 f''(z)\right)}{(1-\lambda)f(z)+\lambda f'(z)}-\square\right)-v\right]\right\}>\gamma .$$

This is equivalent to

$$Re\left\{\frac{vb\left((1-\lambda)f(z)+\lambda f'(z)\right)+a(1+\square)\left(zf'(z)+\lambda z^2 f''(z)\right)-a(\square+V)\left((1-\lambda)f(z)+\lambda f'(z)\right)}{b[(1-\lambda)f(z)+\lambda f'(z)]}\right\}$$

$$= Re\left\{\frac{a(1-v)+b(v-\gamma)z-\sum_{n=2}^{\infty}\left((1-\lambda(n+1))(V+\square+2n)a+(\lambda(n+1)-1)(v-\gamma)b\right)a_n z^{-n}}{bz+\sum_{n=2}^{\infty}b(1-\lambda(n+1))a_n z^{-n}-\sum_{n=1}^{\infty}b(1-\lambda(n+1))b_n z^{-n}}-\frac{\sum_{n=2}^{\infty}\left((1-\lambda(n+1))(v+\square+2n)a+(\lambda(n+1)-1)(v-\gamma)b\right)b_n(\bar{z})^{-n}}{bz+\sum_{n=2}^{\infty}b(1-\lambda(n+1))a_n z^{-n}-\sum_{n=1}^{\infty}b(1-\lambda(n+1))b_n z^{-n}}\right\}\geq 0.$$

This inequality must hold for all $z \in \bar{U}$ and for , $\frac{1}{2}\leq\lambda<1$, $0\leq\gamma<v<1$, $0<\square<1$ and for any $<0|a|<1{,}0<|b|<1$.

Letting $z=r>1$ and a,b are real and positive so that $|a|=a, |b|=b$,we have

$$= Re\left\{\frac{a(1-v)+b(v-\gamma)-\sum_{n=2}^{\infty}\left((1-\lambda(n+1))(V+\square+2n)a+(\lambda(n+1)-1)(v-\gamma)b\right)a_n r^{-n-1}}{b+\sum_{n=2}^{\infty}b(1-\lambda(n+1))a_n r^{-n-1}-\sum_{n=1}^{\infty}b(1-\lambda(n+1))b_n r^{-n-1}}\right.$$

$$\left.-\frac{\sum_{n=2}^{\infty}\left((1-\lambda(n+1))(v+\square+2n)a+(\lambda(n+1)-1)(v-\gamma)b\right)b_n(\bar{r})^{-n-1}}{b+\sum_{n=2}^{\infty}b(1-\lambda(n+1))a_n r^{-n-1}-\sum_{n=1}^{\infty}b(1-\lambda(n+1))b_n r^{-n-1}}\right\}=\frac{A(r)}{B(r)}\geq 0$$

If the condition (6) does not hold , then $A(r)$ is negative for r sufficienty close to 1 . Thus there exists a $z_0=r>1$ for which the quotient $\frac{A(r)}{B(r)}$ is negative .This contradicts that $\frac{A(r)}{B(r)}\geq 0$ and so that the proof is complete.

Extreme points for $WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$ are given in following Theorem .

Theorem (3): Let the function f be given by (3) . Then $f \in WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$if and only if

$$f(z)=\sum_{n=1}^{\infty}\left(x_n h_n(z)+y_n g_n(z)\right) ,z\in\bar{U} \ (7)$$

where $h_1(z)=z$,

$$h_n(z)=z-\frac{|a|(1-v)+|b|(v-\gamma)}{\left(1-\lambda(n+1)\right)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}z^{-n}\ (n=2,3,..)$$

and

$$g_n(z)=z-\frac{|a|(1-v)+|b|(V-\gamma)}{\left(1-\lambda(n+1)\right)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}(\bar{z})^{-n}\ (n=1,2,\dots)$$

$\sum_{n=1}^{\infty}(x_n+y_n)=1$, $(x_n\geq 0, y_n\geq 0)$.

In particular , the extreme points of $WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$ are h_n and g_n.

Proof: Assume that f can be expressed by (7) . Then , we have

$$f(z)=\sum_{n=1}^{\infty}\left(x_n h_n(z)+y_n g_n(z)\right)$$

$$=\sum_{n=1}^{\infty}(x_n+y_n)z-\sum_{n=2}^{\infty}\frac{|a|(1-v)+|b|(v-\gamma)}{\left(1-\lambda(n+1)\right)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}x_n z^{-n}$$

$$-\sum_{n=1}^{\infty}\frac{|a|(1-v)+|b|(v-\gamma)}{\left(1-\lambda(n+1)\right)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}y_n(\bar{z})^{-n}$$

$$=z-\sum_{n=2}^{\infty}\frac{|a|(1-v)+|b|(v-\gamma)}{\left(1-\lambda(n+1)\right)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}x_n z^{-n}$$

$$-\sum_{n=1}^{\infty}\frac{|a|(1-v)+|b|(v-\gamma)}{\left(1-\lambda(n+1)\right)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}y_n(\bar{z})^{-n}$$

Therefore

$$\sum_{n=2}^{\infty}\left(1-\lambda(n+1)\right)(v+\square+2n)|a|$$

$$+(\lambda(n+1)-1)(v-\gamma)|b|)\frac{|a|(1-v)+|b|(v-\gamma)}{\left(1-\lambda(n+1)\right)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}x_n$$

$$+\sum_{n=1}^{\infty}\big(1-\lambda(n+1)\big)(V+m+2n)|a| + (\lambda(n+1)-1)(V-\gamma)|b|\big)\frac{|a|(1-v)+|b|(v-\gamma)}{\big(1-\lambda(n+1)\big)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}y_n$$

$$=\big(|a|(1-v)+|b|(v-\gamma)\big)\left(\sum_{n=1}^{\infty}(x_n+y_n)-x_1\right)$$

$$=\big(|a|(1-v)+|b|(v-\gamma)\big)(1-x_1)\le\big(|a|(1-v)+|b|(v-\gamma)\big)$$

So $f\in WR_{\overline{H}}(a,b;\lambda,\gamma;v,\square)$

Conversely , Let $f\in WR_{\overline{H}}(a,b;\lambda,\gamma;v,\square)$, by putting

$$x_n=\frac{\big(1-\lambda(n+1)\big)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}{|a|(1-v)+|b|(v-\gamma)}a_n\ ,(n=2,3,\dots)$$

and

$$y_n=\frac{\big(1-\lambda(n+1)\big)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}{|a|(1-v)+|b|(v-\gamma)}b_n,(n=1,2,\dots).$$

We define $x_1=1-\sum_{n=2}^{\infty}x_n-\sum_{n=1}^{\infty}y_n$.

Then , note that $0\le x_n\le 1\ (n=2,3,\dots),0\le y_n\le 1\ (n=1,2,\dots)$.

Hence ,

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^{-n}-\sum_{n=1}^{\infty}b_n(\bar{z})^{-n}$$

$$=z-\sum_{n=2}^{\infty}\frac{|a|(1-v)+|b|(v-\gamma)}{\big(1-\lambda(n+1)\big)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}x_nz^{-n}$$

$$-\sum_{n=1}^{\infty}\frac{|a|(1-v)+|b|(v-\gamma)}{\big(1-\lambda(n+1)\big)(v+\square+2n)|a|+(\lambda(n+1)-1)(v-\gamma)|b|}y_n(\bar{z})^{-n}$$

$$=z-\sum_{n=2}^{\infty}\big(z-h_n(z)\big)x_n-\sum_{n=1}^{\infty}\big(z-g_n(z)\big)y_n$$

$$=\left(1-\sum_{n=2}^{\infty}x_n-\sum_{n=1}^{\infty}y_n\right)z+\sum_{n=2}^{\infty}x_nh_n(z)+\sum_{n=1}^{\infty}y_ng_n(z)$$

$$=x_1h_1(z)+\sum_{n=2}^{\infty}x_nh_n(z)+\sum_{n=1}^{\infty}y_ng_n(z)$$

$$=\sum_{n=1}^{\infty}\big(x_nh_n(z)+y_ng_n(z)\big)$$

That is the required representation

Now , we show $WR_{\overline{H}}(a,b;\lambda,\gamma;v,\square)$ is closed under Convex Combination of this members .

Theorem (4): The class $WR_{\overline{H}}(a,b;\lambda,\gamma;v,\square)$ is closed under Convex Combination .

Proof: For $j=1,2,3,\dots$, let $f_j\in WR_{\overline{H}}(a,b;\lambda,\gamma;v,\square)$, where f_j is given

$$f_j(z)=z+\sum_{n=2}^{\infty}a_{n,j}z^{-n}-\sum_{n=1}^{\infty}b_{n,j}(\bar{z})^{-n}.$$

Then by (4) , we have

$$\sum_{n=2}^{\infty}\Big(\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\Big)a_{n,j} + \sum_{n=1}^{\infty}\Big(\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\Big)b_{n,j} \le |a|(1-v)+|b|(v-\gamma)\ (8)$$

For $\sum_{j=1}^{\infty}t_j=1\ ,0\le t_j\le 1$, the Convex Combination of f_j may be written as

$$\sum_{j=1}^{\infty}t_jf_j(z)=z+\sum_{n=2}^{\infty}\left(\sum_{j=1}^{\infty}t_ja_{n,j}\right)z^{-n} - \sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}t_jb_{n,j}\right)(\bar{z})^{-n}.$$

Then , by (4) , we have

$$\sum_{n=2}^{\infty}\Big(\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\Big)\left(\sum_{j=1}^{\infty}t_j\,a_{n,j}\right) + \sum_{n=1}^{\infty}\Big(\big(1-\lambda(n+1)\big)(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|\Big)\left(\sum_{j=1}^{\infty}t_j\,b_{n,j}\right)$$

$$= \sum_{j=1}^{\infty} t_j \left\{ \sum_{n=2}^{\infty} \Big(\big(1-\lambda(n+1)\big)(v+?+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b| \Big) a_{n,j} + \sum_{n=1}^{\infty} \Big(\big(1-\lambda(n+1)\big)(v+?+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b| \Big) b_{n,j} \right\}$$

$$\leq \sum_{j=1}^{\infty} t_j [|a|(1-v)+|b|(v-\gamma)] = |a|(1-v)+|b|(v-\gamma),$$

therefore

$$\sum_{j=1}^{\infty} t_j f_j(z) \in WR_H(a,b;\lambda,\gamma;v,?),$$

This complete this proof .

The distortion bound for functions in $WR_{\overline{H}}(a,b;\lambda,\gamma;v,?)$ are given by following theorem

Theorem (5): Let $\in WR_{\overline{H}}(a,b;\lambda,\gamma;v,?)$. Then for $|z|=r<1$, we have

$$|f(z)| \geq (1-b_1)r^1 - \frac{[|a|(1-v)+|b|(v-\gamma)](1-b_1)}{(1-3\lambda)(4+v+?)+(3\lambda-1)(v-\gamma)} r^{-2} \quad (9)$$

and

$$|f(z)| \leq (1-b_1)r^1 + \frac{[|a|(1-v)+|b|(v-\gamma)](1-b_1)}{(1-3\lambda)(4+v+?)+(3\lambda-1)(v-\gamma)} r^{-2} \quad (10)$$

Proof: Assume that $WR_{\overline{H}}(a,b;\lambda,\gamma;v,?)$. Then by (4) , we get

$$|f(z)| = \left| z + \sum_{n=2}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n (\bar{z})^{-n} \right|$$

$$\geq (1-b_1)r^{-1} - \sum_{n=2}^{\infty} (a_n+b_n) r^{-n}$$

$$\geq (1-b)r^{-1} - \sum_{n=2}^{\infty} (a_n+b_n)\, r^{-2}$$

$$= (1-b_1)r^{-1} - \frac{1}{(1-3\lambda)(4+v+)?+(3\lambda-1)(v-\gamma)} \sum_{n=2}^{\infty} (1-3\lambda)(4+v+?)+(3\lambda-1)(v-\gamma)(a_n+b_n)$$

$$\geq (1-b_1)r^{-1} - \frac{1}{(1-3\lambda)(4+v+?)+(3\lambda-1)(v-\gamma)} \sum_{n=2}^{\infty} (1-\lambda(n+1)(v+?+2n)+(\lambda(n+1)(v-\gamma)(a_n+b_n)$$

$$\geq (1-b_1)r^{-1} - \frac{1}{(1-3\lambda)(4+v+?)+(3\lambda-1)(v-\gamma)} [[|a|(1-v)+|b|(v-\gamma)] - [|a|(1-v)+|b|(v-\gamma)]b_1]r^{-2}$$

$$= (1-b_1)r^{-1} - \frac{[|a|(1-v)+|b|(v-\gamma)](1-b_1)}{(1-3\lambda)(4+v+?)+(3\lambda-1)(v-\gamma)} r^{-2}$$

Relation (10) can be proved by using the similar statement . So, the proof is Complete .

Theorem (6): The class $WR_{\overline{H}}(a,b;\lambda,\gamma;v,?)$ is Convex set .

Proof: Let the function $f_j (j=1,2)$ be in class $WR_H(a,b;\lambda,\gamma;v,?)$.

It is sufficient to show that the function h defined by

$$h(z) = (1-t)f_1(z) + tf_2(z) \, , (0 \leq t < 1)$$

it is in the class $WR_{\overline{H}}(a,b;\lambda,\gamma;v,?)$, where

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^{-n} - \sum_{n=1}^{\infty} b_{n,j} (\bar{z})^{-n} \, , (j=1,2)$$

Since for $0 \leq t \leq 1$,

$$h(z) = z + \sum_{n=2}^{\infty} \big((1-t)a_{n,1} - t a_{n,2} \big) z^{-n} - \sum_{n=1}^{\infty} \big((1-t)b_{n,1} - t b_{n,2} \big) (\bar{z})^{-n} .$$

In view of Theorem (2) , we have

$$\sum_{n=2}^{\infty} \Big(\big(1-\lambda(n+1)\big)(v+?+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b| \Big) \big((1-t)a_{n,1} - t a_{n,2} \big)$$

$$+ \sum_{n=1}^{\infty} \Big(\big(1-\lambda(n+1)\big)(v+?+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b| \Big) \big((1-t)b_{n,1} - t b_{n,2} \big)$$

$$= (1-t) \left(\sum_{n=2}^{\infty} \Big(\big(1-\lambda(n+1)\big)(v+?+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b| \Big) a_{n,1} + \sum_{n=1}^{\infty} \Big(\big(1-\lambda(n+1)\big)(v+?+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b| \Big) b_{n,1} \right)$$

$$+ t \left(\sum_{n=2}^{\infty} \Big(\big(1-\lambda(n+1)\big)(v+?+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b| \Big) a_{n,2} + \sum_{n=1}^{\infty} \Big(\big(1-\lambda(n+1)\big)(v+?+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b| \Big) b_{n,2} \right)$$

$$\leq (1-t)\big(|a|(1-v) + |b|(v-\gamma) + t(|a|(1-v) + |b|(v-\gamma)\big) = |a|(1-v) + |b|(v-\gamma)$$

Hence $h \in WR_{\overline{H}}(a,b;\lambda,\gamma;v,?)$

We define the Convolution two harmonic function f and F by

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^{-n} - \sum_{n=1}^{\infty} b_n B_n (\bar{z})^{-n} \text{ , (11)}$$

where

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n (\bar{z})^{-n} \text{ ,}$$

and

$$F(z) = z^p + \sum_{n=2}^{\infty} A_n z^{-n} - \sum_{n=1}^{\infty} B_n (\bar{z})^{-n} \text{ .}$$

Theorem (7): For $0 \le \beta \le \gamma < 1$. Let $f \in WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square)$ and $F \in WR_{\bar{H}}(a,b;\lambda,\gamma,\beta,\square)$. Then $f * F \in WR_{\bar{H}}(a,b;\lambda,\gamma;v,\square) \subset WR_{\bar{H}}(a,b;\lambda,\gamma;\beta,\square)$

Proof: Let the Convolution $(f * F)$ be of form (10) . Then we want to prove that the Coefficient of $(f * F)$ Satisfy the Condition of Theorem (2) .

Since $f \in WR_{\bar{H}}(a,b;\lambda,\gamma,v,\square)$ and $\in WR_{\bar{H}}(a,b;\lambda,\gamma,\beta,\square)$. Then by Theorem (2) , we have

$$\sum_{n=2}^{\infty} \frac{(1-\lambda(n+1)) + (v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|}{|a|(1-v)+|b|(v-\gamma)} a_n + \sum_{n=1}^{\infty} \frac{(1-\lambda(n+1)) + (v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|}{|a|(1-v)+|b|(v-\gamma)} b_n \le 1$$

and

$$\sum_{n=2}^{\infty} \frac{(1-\lambda(n+1)) + (v+\square+2n)|a| + (\lambda(n+1)-1)(v-\beta)|b|}{|a|(1-v)+|b|(v-\beta)} A_n + \sum_{n=1}^{\infty} \frac{(1-\lambda(n+1)) + (v+\square+2n)|a| + (\lambda(n+1)-1)(v-\beta)|b|}{|a|(1-v)+|b|(v-\beta)} B_n \le 1$$

From (4) , we obtain the following inequalities

$$A_n \le \frac{|a|(1-v)+|b|(v-\beta)}{(1-\lambda(n+1)) + (v+\square+2n)|a| + (\lambda(n+1)-1)(v-\beta)|b|} \quad (n = 2,3,\dots)$$

$$B_n \le \frac{|a|(1-v)+|b|(v-\beta)}{(1-\lambda(n+1)) + (v+\square+2n)|a| + (\lambda(n+1)-1)(v-\beta)|b|} \quad (n = 1,2,\dots)$$

Therefore ,

$$\sum_{n=2}^{\infty} \frac{(1-\lambda(n+1))(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|}{|a|(1-v)+|b|(v-\gamma)} a_n A_n + \sum_{n=1}^{\infty} \frac{(1-\lambda(n+1))(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|}{|a|(1-v)+|b|(v-\gamma)} b_n B_n$$

$$= \frac{(1-2\lambda)(2+v+\square)|a| + (2\lambda-1)(v-\gamma)|b|}{|a|(1-v)+|b|(v-\gamma)} b_1 B_1 + \sum_{n=2}^{\infty} \frac{(1-\lambda(n+1))(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|}{|a|(1-v)+|b|(v-\gamma)} a_n A_n + \sum_{n=1}^{\infty} \frac{(1-\lambda(n+1))(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|}{|a|(1-v)+|b|(v-\gamma)} b_n B_n$$

$$< \frac{[(1-2\lambda)(2+v+\square)|a| + (2\lambda-1)(v-\gamma)|b|][|a|(1-v)+|b|(v-\beta)]}{[|a|(1-v)+|b|(v-\gamma)][(1-\lambda(n+1)(v+\square+2)|a| + (\lambda(n+1)-1(v-\beta)|b|]} + \sum_{n=2}^{\infty} \frac{[(1-\lambda(n+1))(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|][|a|(1-v)+|b|(v-\beta)]}{[|a|(1-v)+|b|(v-\gamma)][(1-\lambda(n+1))(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\beta)|b|]} a_n + \sum_{n=2}^{\infty} \frac{[(1-\lambda(n+1))(V+m+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|][|a|(1-v)+|b|(v-\beta)]}{[|a|(1-v)+|b|(v-\gamma)][(1-\lambda(n+1))(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\beta)|b|]} b_n$$

$$= \sum_{n=2}^{\infty} \frac{[(1-\lambda(n+1))(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|][|a|(1-v)+|b|(v-\beta)]}{[|a|(1-v)+|b|(v-\gamma)][(1-\lambda(n+1))(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\beta)|b|]} a_n + \sum_{n=1}^{\infty} \frac{[(1-\lambda(n+1))(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|][|a|(1-v)+|b|(v-\beta)]}{[|a|(1-v)+|b|(v-\gamma)][(1-\lambda(n+1))(v+\square+2n)|a| + (\lambda(n+1)-1)(v-\beta)|b|]} b_n$$

$$\leq \sum_{n=2}^{\infty} \frac{(1-\lambda(n+1))(v+ ⍰ +2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|}{|a|(1-v)+|b|(v-\gamma)} a_n$$

$$+ \sum_{n=1}^{\infty} \frac{(1-\lambda(n+1))(v+ ⍰ +2n)|a| + (\lambda(n+1)-1)(v-\gamma)|b|}{|a|(1-v)+|b|(v-\gamma)} b_n \leq 1$$

Then $(f * F) \in WR_{\overline{H}}(a,b;\lambda,\gamma;v, ⍰) \subset WR_{\overline{H}}(a,b;\lambda,\gamma;\beta, ⍰)$.

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