Operators on Finite Dimensional Inner Product Space

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Abstract: ***Operators play a pivotal role in pure and applied branch of mathematical sciences. Operators are specially integral part of algebra and analysis. In this article we are going to discuss some basic operators on finite dimensional inner product spaces and we also emphasized the application oriented treatment of some of these operators.***

Keywords: Self Adjoint operator, Unitary operator, Orthogonal and normal operators

1. Introduction

In this article we discus those functions that preserve the structure of thee underlying space In particular, linear operators preserve the operation of vector addition and multiplication and isomorphism preserve all the vector space structure .Natural to consider those linear operator T on inner product space that preserve length.

2. Self Adjoint Operator on Inner Product Space

Definition

Let v be a finite dimensional inner product space and T to be a linear operator on V .We call T is **selfadjoin**t operator on V if$T = T^*$; an n x n matrix A (**Real or Complex**) is selfadjoint if$A = A^*$.

Properties Let V be a finite dimensional inner product space and T be a linear operators on V .We call T is **selfadjoint** operator on V then following axioms are true**:**

(a) Eigen value of T is real.

Proof: we know that $\lambda < v, v > = \bar{\lambda} < v, v >$,

$\lambda < v, v > = < \lambda v, v > =$

$=< T(v), v >= < v, T^*(v) >$

$=< v, T(v) > \; = < v, \lambda v >$

$=\bar{\lambda} < v, v >$

But $< v, v > \neq 0; Hence\ \lambda = \bar{\lambda}$ and so λ is real.

(b) Let V be an inner product space (**Real**) then characteristic polynomial of T splits.

Proof: Let dim (v) = n and β be a basis of V and A=$[T]_\beta$, then A is selfadjoint, Let T be linear operator on C^n, such that $T_A(x) = Ax$ for all x∈C^n.by (a) eigen value of T_A is real. We know that characteristic polynomial of T_A splits into factors of the form T-λ I , (λ is real). But T_A has same characteristic polynomial as A which has same characteristic polynomial as T. Therefore the characteristic polynomial of T splits.

3. Unitary and Orthogonal Operator

Definition

Let V be a finite dimensional inner product space over field F and T be a linear operator on V. If $\|T(v)\|=\|v\|$ for all v ∈V. We call T is unitary operator if field is complex and an orthogonal if field F is real.

In infinite dimensional cases an operator satisfying the preceding norm requirement is called i**s**ometry.

Properties

Let V be a finite dimensional inner product space and T be a linear operator on V then following postulates are equivalent on an operator T.

(a) $TT^* = T^*T = I\ i.e. T^* = T$

(b) $<T(v), T(w) >=< v, w >$ for all v, w ∈ V

(c) If β is orthogonal basis foe V such that T(β) orthogonal basis for V.

Proof: -

We first prove that **(a)=>(b);**

Let v, w ∈V,

$< v, w >=< T(v), T(w) >$

$=< TT^*v, w >$

$=< T(v), T(w) >$

(b)=> (c)

Let x∈ V then $\|T(v)\|=\|v\|$.

Let β= $\{v_1, v_2, \dots \dots \dots v_n\}$ be the orthogonal basis for V So $T(\beta) = \{ T(v_1), T(v_2), \dots \dots \dots \dots . T(v_v)\}$,

It follows that $< T(v_i), T(v_j) > = < v_i, v_j > = \delta_{ij}$.

Therefore $T(\beta)$ is an orthogonal basis for V.

Definition of Unitary matrix

A complex matrix A for which $A^* = A^{-1}$ or equivalently $AA^* = A^*A = I$, is called a unitary matrix.

Definition of orthogonal matrix

A matrix A for which $A^t = A^{-1}$ *or* equivalently $AA^t = A^tA = I$ is called an orthogonal matrix.

Example:-

Suppose A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a unitary matrix.

Then $AA^* =$ I,

$AA^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} |a|^2+|b|^2 & a\bar{c}+b\bar{d} \\ \bar{a}c+d\bar{b} & |c|^2+|d|^2 \end{pmatrix}$
$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$

Thus $|a|^2+|b|^2=1, |c|^2+|d|^2=1$ and $a\bar{c}+b\bar{d}=0$. The row of a form an orthogonal set, similarly $AA^*=$ I, forces the column of A to form an orthogonal set.

4. Applications

4.1 Rigid Motion

Rigid motion is an application to characterize for a transformation is that it preserves distance of rigid motion of a finite dimensional inner product space, such as a transformation which does not change the shape of figure under its action.

Definition

Let V be a inner product space .A function $f{:}V \to V$ is called a rigid motion if

$\|f(v)-f(w)\| = \|v-w\|$ for all v,w ε V

Properties

Let $f{:}V \to V$ be a rigid motion on a finite dimensional real inner product space V, then there exist a unique orthogonal operator T on V and a unique translation g on V such that $f = goT$.

Proof- Let $T{:}V \to V$ is defined by $T(v) = f(v) - f(0)$ for all v∈V.
We know that T is orthogonal operator then$f = goT$, where g is the translation by $f(0)$.Here T is the composite of f and the translation by $f(v) - f(0)$; thus T is rigid motion.

For any v ∈V.
$\|T(v)\|^2 = \|f(v)-f(0)\|^2 = \|v-0\|^2 = \|v\|^2$.

For any v,w ∈V.
$\|T(v)-T(w)\|^2$
$= \|T(v)\|^2 - 2 < T(v),T(v) > +\|T(w)\|^2$
$= \|v\|^2 - 2 < T(v),T(v) > +\|w\|^2$
And $\|v-w\|^2 = \|v\|^2 - 2 < v,w > +\|w\|^2$
But $\|T(v)-T(w)\|^2 = \|v-w\|^2$;
and $< T(v),T(v) > = < v,w >$ for all v,w ∈V

Now in the position to show that T is a linear transformation
Let v,w ∈V and a ∈V then
$\|T(v+aw)-T(v)-aT(w)\|^2$
$= \|[T(v+aw)-T(v)]-aT(w)\|^2$
$= \|T(v+aw)-T(v)\|^2 + a^2\|T(w)\|^2 - 2a <$
$T(v+aw)-T(v),T(w) >$.

$= \|(v+aw)-v\|^2 + a^2\|T(w)\|^2 - 2a < T(v+aw) -$
$T(v),T(w) >$
$= a^2\|w\|^2 + a^2\|w\|^2 - \quad 2a[< v+aw,w >, -<$
$v,w >]$
$=2a^2\|w\|^2 - 2a\left[< v,w > + a\|w\|^2 - < v,w > \right] = 0$

Thus T is linear

$T(v+aw) = T(v) + aT(w)$
and T also preserve Orthogonal operator

To prove uniqueness

Let v_0 and v are in V and T & U are orthogonal operator in V, Putting $v=0$ then $u_0 = v_0$ and thus the translation is unique. Reduces $T(v) = U(v)$ for all $v \in V$
Thus T=U.

4.2 Conic Section

A conic section is great details, and introduces a novel numerical calculation method for geometric algebra that incorporates the notion of random algebra variables .A particularly interesting contribution is the introduction of the conic section which refers to the geometric algebra of over the vector space of reduced symmetric matrices

Consider a Quadratic equation
$ax^2 + 2bxy + cy^2 + dx + ey + f = 0 \ldots\ldots(A)$
we get various conic section for particular choice of coefficient as circle ,ellipse, parabola and hyperbola for example a=c=1,b=d=e=0 & f= -1,we get a circle x^2+y^2=1 whose centre at origin .If b=0 then x-y term is absent the equation (A) is written as

for example x^2+2x+y^2+4y+2=0, is rewritten as $(x+1)^2+(y+2)^2=3$,which is a circle of radius $\sqrt{3}$ and centre (-1,-2) in x-y coordinate system.

If the transformation of coordinate change by $(x,y)\to(\acute{x}+\acute{y})$ where $\acute{x} = x+1$ &$\acute{y}$ =y+2 then equation change as $\acute{x}^2+\acute{y}^2$=3 this change allow us in the variable the eliminate the x and y term.

Consider an expression
$ax^2 + 2bxy + cy^2 = 0$ …………………(B)
is called associated quadratic form of (A)
We obtains A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and X = $\begin{pmatrix} x \\ y \end{pmatrix}$
In this way (B) can be written as
$(X^t AX) = (AX, X)$.
Consider the quadratic equation
$2x^2 - 4xy + 5y^2 - 36 = 0$
Associated quadratic form is $2x^2 - 4xy + 5y^2$
Can be written as A=$\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$
The eigen values of A is 1 and 6 with associated eigen vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ these vector is orthogonal.
the corresponding orthogonal basis of eigenvectors
$\beta=\left\{\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}\right\}$
determine new axis x' and y'
$P=\left\{\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}\right\} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
Then $P^t AP = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$

In the way X=PX',

$$x = \frac{2}{\sqrt{5}}x' - \frac{1}{\sqrt{5}}y',$$

$$y = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'$$

The original equation change in the another equation to related new coordinate system$(x')^2 + 6(y')^2$=36 represents an ellipse. The dimension- independent representation of geometric entities and reflection also has the effect that a transformation operator can be applied to any element in the algebra in any dimension, be it a geometric entity or another transformation.

5. Conclusion

Operators are so important that algebra and analysis are incomplete without their mention. They provide a strong tool for researcher who are doing research in algebra particularly in linear algebra and in analysis specially in functional analysis**.**

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