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Mildly β-Normal Spaces and rgβ-Continuous Functions

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Abstract: ***The aim this paper is to introduce and study a new class of spaces, called mildly β-normal spaces. The relationship among normal, almost normal, quasi normal, mildly normal, π-normal spaces and their generalizations are investigated. Moreover, we introduce rgβ-continuous functions. Utilizing rgβ-continuity, we obtain characterization and preservation theorems for mildly β-normal spaces.***

Keywords: β-open, β-normal, mildly β-normal spaces, gβ-closed and rgβ-closed function.

2010 Mathematics subject classification: 54D10, 54D15, 54A05, 54C08.

1. Introduction

The notion of quasi normal space was introduced by Zaitsev [**25**]. The concept of almost normality was introduced by Singal and Arya [**20**]. The notion of mildly normal space was introduced by Shchepin [**19**] and Singal and Singal [**22**] independently. Nour [**4**] introduced a weaker form of normality, called p-normality and obtained their properties. Mahmoud et al. [7] introduced the notion of β-normal spaces and obtained their characterizations and preservation theorems. Dontchev and Noiri [**4**] introduce the notion of πg-closed sets as a weak form of g-closed sets due to Levine [**6**]. By using πg–closed sets, Dontchev and Noiri [**8**] obtained a new characterization of quasi normal spaces. π–normal topological space was introduced by Kalantan [**5**]. M. C. Sharma and Hamant Kumar [**17**] introduced a weaker form of normality, called πβ-normality and obtained their properties. The notion of quasi β-normal and mildly β–normal spaces were introduced by M. C. Sharma and Hamant Kumar [**16**]. Thabit and Kamaraulhaili [**24**] introduced a new type of normality, called πp–normality and obtained their characterizations. The concept of quasi p-normality was introduced by Thabit and Kamaraulhaili [**23**].The notion of almost p-normal and mildly p-normal spaces were introduced by G. B. Navalagi [**10**].The notion of almost β-normal space was introduced by M. C. Sharma and Hamant Kumar [**18**].

2. Preliminaries

Throughout this paper, spaces (X, τ), (Y, σ), and (Z, γ) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and interior of A are denoted by cl(A) and int(A) respectively. A subset A is said to be **regular open** (resp. **regular closed**) if A = int(cl(A)) (resp. A = cl(int(A)). The finite union of regular open sets is said to be **π-open**. The complement of a π-open set is said to be **π-closed.** A is said to be **β-open** [1] if A ⊂ cl(int(cl(A))), **preopen** [9] (briefly **p-open**) if A ⊂ int(cl(A)). The complement of a β–open (resp. p-open) set is said to be **β–closed** [**1**] (resp. **p-closed** [**9**]). The intersection of all β–closed (resp. p-closed) sets containing A is called **β–closure** [**2**] (resp. **p-closure** [**9**]) of A, and is denoted by βcl(A) (resp. pcl(A). The **β-Interior** [**2**] of A, denoted by βint(A), is defined as union of all β-open sets contained in A.

2.1 Definition

A subset A of a space (X, τ) is said to be

1) **generalized closed** (briefly **g-closed**) [**6]** if cl(A) ⊂ U whenever A ⊂U and U∈ τ.
2) **regular generalized closed (**briefly **rg-closed [14])** if cl(A) ⊂ U whenever A⊂ U and U is regular open in X .
3) **generalized preclosed** (briefly **gp-closed**) [**8**]) if pcl(A) ⊂ U whenever A ⊂ U and U is open in X.
4) **regular generalized preclosed** (briefly r**gp-closed**) [**11**]) if pcl(A) ⊂ U whenever A ⊂ U and U is regular open in X.
5) **generalized β-closed** (briefly **gβ-closed**) [**3**]) if βcl(A) ⊂ U whenever A ⊂ U and U is open in X.
6) **regular generalized β- closed** (briefly **rgβ-closed**) [**15**]) if βcl(A) ⊂ U whenever A ⊂ U and U is regular open in X.
7) **g-open** (resp**. rg-open**, **gp-open**, **rgp-open**, **gβ-open**, r**gβ-open**) if the complement of A is g–closed (resp. rg-closed, gp-closed, rgp-closed, gβ-closed, rgβ–closed).

2.2 Remark

We have the following implications for the properties of subsets:

closed ⇒	g-closed ⇒	rg-closed
⇓	⇓	⇓
p-closed ⇒	gp-closed ⇒	gpr-closed
⇓	⇓	⇓
β-closed ⇒	gβ-closed ⇒	rgβ-closed

Where none of the implications is reversible as can be seen from the following examples:

2.3 Example

Let X = {a, b, c} and τ = {∅, {a}, X}. Then A= {b} is g–closed but not closed.

2.4 Example

Let X = {a, b, c, d, e,} and τ = {∅, {a, b}, {c, d}, {a, b, c, d}, X}. Then A = {a} is gpr-closed as well as rgβ-closed. But it is not rg-closed.

2.5 Example

Let X = {a, b, c,} and τ = {∅, {a}, {b}, {a, b}, X}. Then A = {a, b} is gpr-closed as well as rgβ –closed. But it is neither gp-closed nor p-closed.

2.6 Example

Let X = {a, b, c, d} and τ = {∅, {c}, {d}, {a, c}, {c, d}, {a, c, d}, X}.Then, A = {a}is p-closed as well as β-closed. But it is not rg-closed.

2.7 Example

Let X = {a, b, c} and τ = {∅, {a}, {a, b}, X}. Then A = {b} is gβ- closed as well as rgβ-closed. But it is neither g-closed nor closed.

2.8 Example

Let X = {a, b, c} and τ = {∅, {a}, {b}, {a, b}, X} Then A = {a} is β-closed as well as gβ-closed but it is not closed.

2.9 Lemma

A subset A of a space X is rgβ-open if and only if F ⊂ βint(A) whenever F is a regular closed and F ⊂ A.

3. Mildly β-Normal Spaces

3.1 Definition

A topological space X is said to be **quasi β-normal** [16] (resp. **quasi-normal** [25], **quasi p-normal** [23]) if for every pair of disjoint π-closed subsets H, K, there exist disjoint β-open (resp. open, p-open) sets U, V of X such that H ⊂ U and K ⊂ V.

3.2 Definition

A topological space X is said to be **mildly β-normal** [20] (resp. **mildly-normal** [19, 22], **mildly p-normal** [10]) if for every pair of disjoint regular closed subsets H, K, there exist disjoint β-open (resp. open, p-open) sets U, V of X such that H ⊂ U and K ⊂ V.

3.3 Definition

A topological space X is said to be **πβ-normal [17] (**resp. **π-normal** [5], **πp-normal [24])** if for any two disjoint closed subsets A and B of X, one of which is π–closed, there exist disjoint β–open (resp. open, p-open) sets U and V of X such that A ⊂ U and B ⊂ V.

3.4 Definition

A topological space X is said to be **almost β-normal [18]** (resp. **almost normal** [20], **almost p-normal** [10]) if for any two disjoint closed subsets A and B of X, one of which is regular closed, there exist disjoint β-open (resp. open, p-open) sets U and V of X such that A ⊂ U and B ⊂ V.

3.5 Definition

A topological space X is said to be **β–normal** [7] (resp. **p-normal [13]**) if for every pair of disjoint closed subsets A, B of X, there exist disjoint β–open (resp. p-open) sets U, V of X such that A ⊂ U and B ⊂ V.

3.6 Example

Let X = {a, b, c} and τ = {∅, {a}, {a, b}, {a, c}, X}. Then (X, τ) is almost p–normal space, but it is not p-normal, since the pair of disjoint closed sets {b} and {c} have no disjoint p-open sets containing them.

3.7 Example

Let X = {a, b, c, d} and τ = {∅, {b, d}, {a, b, d}, {b, c, d}, X}. The pair of disjoint closed subsets of X are A = {a} and B = {c}. Also U = {a, b} and V = {c, d} are β–open sets such that A ⊂ U and B ⊂ V. Hence X is β-normal but it is not normal.

3.8 Example

Let X = {a, b, c} and τ = {∅, {a}, {a, b}, {a, c}, X}. Then (X, τ) is quasi p-normal space, but it is not p-normal.

3.9 Example

Let X = {a, b, c, d} and τ = {∅, {b}, {d}, {b, d}, {a, b, d}, {b, c, d}, X}. The pair of disjoint closed subsets of X are A = {a} and B = {c}. Also U = {a, b} and V = {c, d} are p-open sets such that A ⊂ U and B ⊂ V. Hence (X, τ) is p-normal but it is not normal.

3.10 Example

Let X = {a, b, c} and τ = {∅, {a}, {a, b}, {a, c}, X}. Then (X, τ) is almost normal but it is not normal.

3.11 Example

Let X = {a, b, c} and τ = {∅, {a}, {a, b}, {a, c}, X}. Then (X, τ) is p-normal space because the only closed sets in X are X and ∅. But it is not p-normal, since the pair of disjoint closed sets {b} and {c} have no disjoint p-open sets containing them.

By the definitions and examples stated above, we have the following diagram:

normal $\Rightarrow$ π-normal $\Rightarrow$ almost normal $\Rightarrow$ mildly normal
$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$
p-normal $\Rightarrow$ πp-normal $\Rightarrow$ almost p-normal $\Rightarrow$ mildly p-normal
$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$
β-normal $\Rightarrow$ $\pi\beta$-normal $\Rightarrow$ almost β-normal $\Rightarrow$ mildly β-normal

and

normal $\Rightarrow$ π-normal $\Rightarrow$ quasi-normal $\Rightarrow$ mildly normal
$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$
p-normal $\Rightarrow$ πp-normal $\Rightarrow$ quasi p-normal $\Rightarrow$ mildly p-normal.
$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$
β-normal $\Rightarrow$ $\pi\beta$-normal $\Rightarrow$ quasi β-normal $\Rightarrow$ mildly β-normal.

3.12 Theorem

The following are equivalent for a space X:
1) X is mildly β-normal.
2) For any disjoint regular closed sets A and B, there exist disjoint gβ-open sets U and V such that $A \subset U$ and $B \subset V$
3) For any disjoint regular closed sets A and B of X, there exist disjoint rgβ-open sets U and V such that $A \subset U$ and $B \subset V$.
4) For any regular closed set A and any regular open set V containing A, there exists a gβ-open set U of X such that $H \subset U \subset \beta cl(U) \subset V$.
5) For any regular closed set A and any regular open set V containing A, there exists a rgβ-open set U of X such that $A \subset U \subset \beta cl(U) \subset V$.

Proof

(1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Let X be mildly β-normal space. Let A, B be disjoint regular closed sets of X. By assumption, there exist disjoint β-open sets U, V such that $A \subset U$ and $B \subset V$. Since every β-open set is gβ-open, so, U and V are gβ-open sets such that $H \subset U$ and $K \subset V$.

(2) $\Rightarrow$ (3). Let A, B be two disjoint regular closed sets. By assumption, there exist gβ-open sets U and V such that $A \subset U$ and $B \subset V$. Since gβ-open set is rgβ-open, so, U and V are rgβ-open such that $H \subset U$ and $K \subset V$.

(3) $\Rightarrow$ (4). Let A be any regular closed set and V be any regular open set containing A. By assumption, there exist rgβ-open sets U and W such that $A \subset U$ and $X\text{-}V \subset W$. By **Lemma 2.9**, we get $X\text{-}V \subset \beta int(W)$ and $U \cap \beta int(W) = \varnothing$. Therefore, we obtain $\beta cl(U) \cap \beta int(W) = \varnothing$ and hence $A \subset U \subset \beta cl(U) \subset X\text{-}\beta Int(W) \subset V$.

(4) $\Rightarrow$ (5). Let A be any regular-closed set and V be any regular-open set containing A. By assumption, there exist gβ-open set U of X such that $H \subset U \subset \beta cl(U) \subset V$. Since, every g$\beta$-open set is rg$\beta$-open, there exist rg$\beta$-open sets U of X such that $H \subset U \subset \beta cl(U) \subset V$.

(5) $\Rightarrow$ (1). Let A, B be any two disjoint regular-closed sets of X. Then $H \subset X - K$ and $X - B$ is regular-open. By assumption, there exists rgβ-open set G of X such that $A \subset G \subset \beta cl(G) \subset X - B$. Put $U = \beta int(G)$, $V = K - \beta cl(G)$. Then U and V are disjoint β-open sets of X such that $A \subset U$ and $B \subset V$.

Using **Theorem 3.12,** it is easy to show the following theorem, which is a Urysohn's Lemma version for mild β-normality. A proof can be established by a similar way of the normal case.

3.13 Theorem

A space X is mildly β-normal if and only if for every pair of disjoint regularly closed sets A and B of X, there exists a continuous function f on X into [0, 1], with its usual topology, such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

It is easy to see that the inverse image of a regularly closed set under an open continuous function is regularly closed. We will use that in the next theorem.

3.14 Theorem

Let X is a mildly β-normal space and $f : X \to Y$ is an open continuous injective function. Then f(X) is a softly normal space.

Proof. Let A and B be any two regularly closed subset of f(X) such that $A \cap B = \varnothing$. Then $f^{-1}(A)$ and $f^{-1}(B)$ regularly closed sets of X. Since X is mildly β-normal, there are two disjoint β-open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Since f is 1-1 and open, since every open set is β–open set, result follows.

3.15 Corollary

Mild β-normality is a topological property.

3.16 Definition. A function $f : X \to Y$ is said to be

1) **rgβ-continuous** if $f^{-1}(F)$ is rgβ-closed in X for every closed set F of Y.
2) **β-rgβ-continuous** if $f^{-1}(F)$ is rgβ-closed in X for every β-closed set F of Y.
3) **rgβ-irresolute** if $f^{-1}(F)$ is rgβ-closed in X for every rgβ-closed set F of Y.
4) **rc-preserving** [**12**] (resp. **almost closed** [**21**]) if f(F) is regular closed (resp. closed) in Y for every regular closed set F of X.

3.17 Theorem

If f : X → Y is a β-rgβ-continuous, rc-preserving injection and Y is mildly β-normal then X is mildly β-normal.

Proof. Let A and B be any disjoint regular closed sets of X. Since f is an rc-preserving injection, f(A) and f(B) are disjoint regular closed sets of Y. By mild β-normality of Y, there exist disjoint β-open sets U and V of Y such that f(A) ⊂ U and f(B) ⊂ V. Since f is β-rgβ-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint rgβ-open sets containing A and B respectively. Hence by **Theorem 3.12** X is mildly β-normal.

3.18 Theorem

If f : X → Y is a β-rgβ-continuous, almost closed surjection and Y is β-normal space, then X is mildly β–normal.

Proof. Similar to previous one.

4. Conclusion

In this paper, we introduced a new class of spaces, called mildly β-normal spaces and established their relationships with some weak forms of normal spaces like normal, almost normal, quasi normal, mildly normal, π-normal spaces and their generalizations in topological spaces.

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