

# Perturbation Analysis of Unsaturated Flow in Dry Clay

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**Abstract:** In present study stability analysis, for 1-D unsaturated flow in clay soil, is done by using Richard's equation. This analysis is valid in very dry conditions, in which Richards equation transforms from quasi non-linear to parabolic form. Perturbation stability analysis of simplified  $\theta$ -based equation shows that function consisting of soil water diffusivity, temporal and spatial nodal space, governs the error propagation. In finite difference scheme the analysis provides an insight into effect of this function the i.e., at different values of this function error magnification changes with trend showing linear variation. At low value of function error propagation is insignificant on nearby nodes and diminishes after some nodes. But as magnitude of function increases situation becomes vice-versa.

**Keywords:** Richard's equation, finite difference, perturbation stability

## 1. Introduction

The zone of unsaturation has vital role in various aspects of hydrological cycle like subsurface flow, infiltration and recharge. But its analysis is difficult due to highly non-linear nature of dependent parameters. According to Darcy's law, flux equals the product of the head gradient and the hydraulic conductivity. Therefore, when the hydraulic conductivity is very small; a very large head gradient is required to move even a small amount of water. This results in very steep head gradients at the wetting front.

The  $\theta$ -based RE has shown good performance as compared to  $\psi$ -based equation (Hills et al., 1989) in very dry conditions. But its application is limited to non-saturated regions only.

$\Psi$ -based equation can be used in saturated as well as unsaturated regions. But while applying it in very dry soils required time steps need to be very small in order to control mass balance errors (Celia et al., 1990).

Mixed form of RE use both  $\theta$  and  $\psi$  and shows relatively good results as compared to pressure based (Celia et al., 1990; Clements et al., 1994). It can be applied easily with good efficiency in dry soils also. So in current study Moisture based equation is used as it gives excellent results in dry conditions.

## 2. Methodology

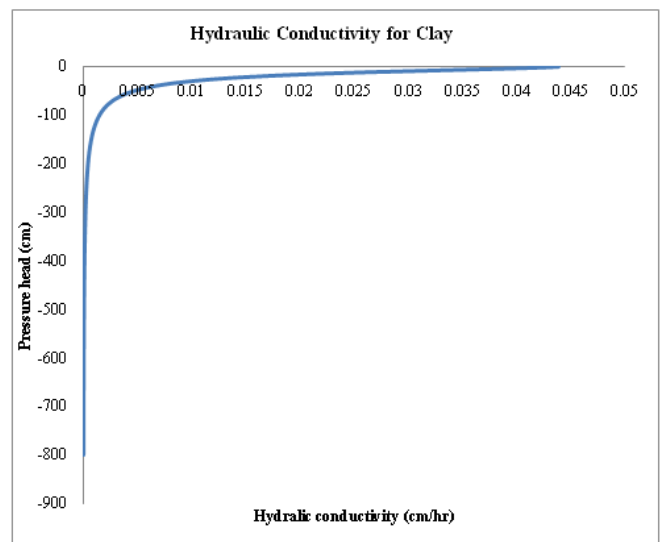
The relationships that govern the flow of water in unsaturated soil are quasi-linear equations of the parabolic type and it becomes elliptical in saturated zone. Since the coefficients in these equations are functions of the dependent variables, exact analytical solutions for specific boundary conditions are extremely difficult to obtain. The governing equation for unsaturated flow (moisture based)

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( \frac{D \partial \theta}{\partial z} \right) - \frac{\partial K}{\partial z} \quad (1)$$

Where  $\theta$  = Soil moisture,  $K$  = hydraulic conductivity,  $D$  =  $(K/C)$  Soil Water Diffusivity,  $C \equiv \frac{d\theta}{dh}$  (specific water

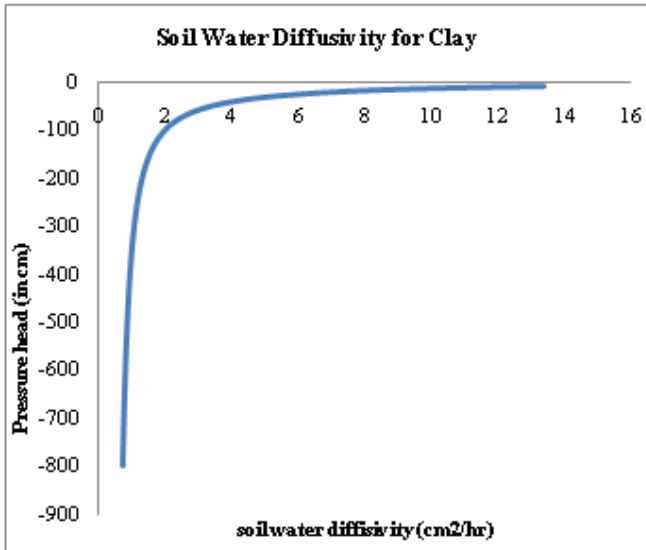
capacity),  $z$  = nodal coordinate in vertical direction and  $t$  = time.

Normally Richard's equation is quasi non-linear in unsaturated zone and has no exact solution except few specific cases. The relation between water content and pressure head depends on type of soils, particle size distribution, bulk density and several other factors. Several empirical formulae are available which give the soil moisture retention curve like Brooks and Corey (1964), Campbell (1974) and Van Genuchten (1980). Typical graphs for soil water hydraulic diffusivity ( $D$ ) and hydraulic conductivity ( $K$ ) are shown below for sandy soil (Celia et al., 1990).



**Figure 1:** General hydraulic conductivity curve for clay soil

For clay in the unsaturated zone,  $K$  is almost constant at low value of  $\psi$ . As the pressure head increases the value of  $K$  also increases but gradient of increment is almost zero in very dry conditions and increases as saturation is approached.



**Figure 2:** General soil water diffusivity curve for clay soil

For claysoil in the unsaturated zone, value of D is almost constant at low value of pressure head. As the pressure head increases, the value of D also increases but rate of increment is very low in unsaturated zone.

### 3. Procedure

We can make following assumption for very low value of  $\theta$  (or say  $\psi$ ) based on Fig.1 and Fig.2 for very low values of pressure head.

1. D ( $\theta$ ) is constant= $D_0$  as D changes insignificantly at very dry conditions.
2.  $\frac{\partial K}{\partial z} = 0$  as K changes insignificantly at very dry conditions.
3. Time steps are very small so that  $\psi$  at nodes under consideration does not change too much.

So, equation 1 takes following parabolic form

$$\frac{\partial \theta}{\partial t} = D_0 \frac{\partial}{\partial z} \left( \frac{\partial \theta}{\partial z} \right) = D_0 \frac{\partial^2 \theta}{\partial z^2} \quad (2)$$

Where  $D_0$  = Diffusivity at  $t=0$ ;  $\theta$  = Soil moisture content; and  $z$  = vertical distance downwards.

Using 2<sup>nd</sup> order central differencing for space derivative and 1<sup>st</sup> order forward differencing for time derivative:

$$\frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} = D_0 \left[ \frac{\theta_{i+1}^n - 2\theta_i^n + \theta_{i-1}^n}{(\Delta x)^2} \right] \quad (3)$$

Where  $D_0$  = Diffusivity at  $t=0$ ;  $i$  = nodal distance points in vertical down direction; and  $n$  = nodal time space.

Now an error of  $\epsilon$  was introduced at node ( $i$ ) and time ( $n$ ).

At point ( $i, n+1$ ) using eq. (4.3)

$$\frac{\theta_i^{n+1} - \epsilon}{\Delta t} = D_0 \left[ \frac{0 - 2\epsilon + 0}{(\Delta x)^2} \right]$$

Where  $D_0 = D$  ( $\text{cm}^2/\text{sec}$ );  $d = D_0 \frac{\Delta t}{[\Delta x]^2}$ ,  $\Delta t$  (sec) = nodal time space;  $\Delta x$  (cm) = nodal distance

In the above equation, it is assumed that  $\theta_{i+1}^n$  and  $\theta_{i-1}^n$  has zero errors.

$$\theta_i^{n+1} = \epsilon(1 - 2d)$$

For stability

$$\left| \frac{\epsilon_i^{n+1}}{\epsilon_i^n} \right| = |1 - 2d| \leq 1 \quad (4)$$

Similarly for ( $i+1, n+1$ ) using eq. (5)

$$\frac{\theta_{i+1}^{n+1} - 0}{\Delta t} = D_0 \left[ \frac{0 - 0 + \epsilon}{(\Delta x)^2} \right]$$

In the immediate above equation, it is assumed that  $\theta_{i+2}^n$  and

$\theta_{i+1}^n$  has zero error

$$\theta_{i+1}^{n+1} = d(\epsilon)$$

For stability

$$\left| \frac{\epsilon_{i+1}^{n+1}}{\epsilon_i^n} \right| = |d| \leq 1 \quad (5)$$

Similarly for ( $i-1, n+1$ ) using eq. (5)

$$\frac{\theta_{i-1}^{n+1} - 0}{\Delta t} = D_0 \left[ \frac{\epsilon - 0 + 0}{(\Delta x)^2} \right]$$

In the above equation, it is assumed that  $\theta_{i-2}^n$  and  $\theta_{i-1}^n$  has zero error

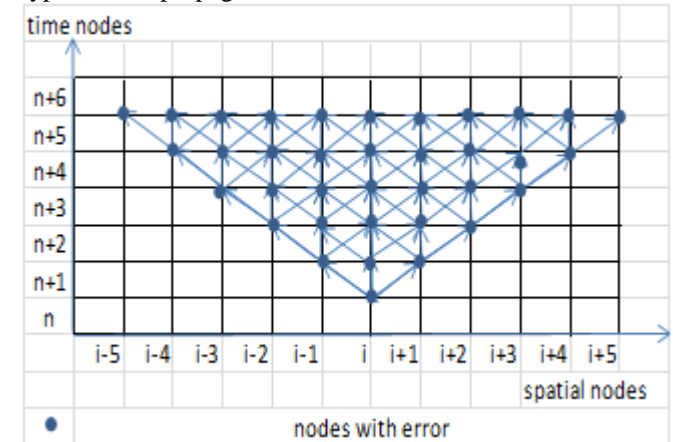
$$\theta_{i-1}^{n+1} = d(\epsilon)$$

For stability

$$\left| \frac{\epsilon_{i-1}^{n+1}}{\epsilon_i^n} \right| = |d| \leq 1 \quad (6)$$

So with each time step error propagates to two surrounding nodes in next time step and so on.

Typical error propagation



**Figure 3:** Typical error propagation in simplified Richard's equation

$i$ : nodal distance points in vertical down direction;  $n$ : nodal time space

### 4. Result and Discussion

While analyzing the error propagation for different values of "d", at the node where error was introduced, it is observed that value of error magnification depends on  $D_0 \frac{\Delta t}{[\Delta x]^2}$ .

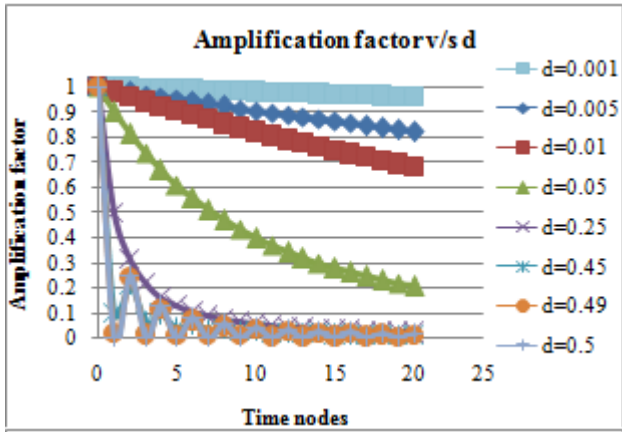


Figure 4: Error Amplification factor for d=0.005, 0.01, 0.05, 0.1, 0.49, 0.50

It is observed that for low value of “d” amplification factor is close to unity at the node at which error was introduced. So value of ‘d’ should not be too low or to say too much wider spatial grid will have error amplification close to unity during the initial wetting period. As the value of “d” increases the amplification factor also decreases sharply and oscillating magnifications are produced for  $d \in [0.45, 0.5]$ . So the value of “d” need to be less than 0.5. So decision to opt for a particular value of d depends on stability as well as processing speed of CPU.

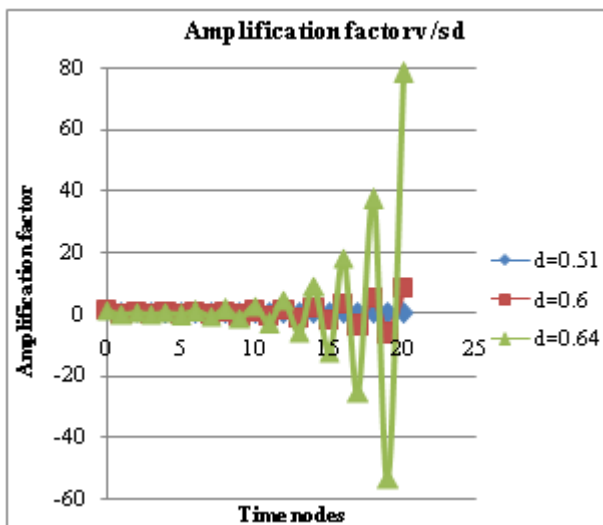


Figure 5: Error Amplification factor for d=0.51, 0.6, 0.64

It is observed that for high value of “d” amplification factor rises exponentially along with oscillations at node at which error was introduced. So “d” should not be too high or to say too much finer spatial grid may produce very high error amplification during the initial wetting period. As the value of “d” increases after 0.5 the amplification factor also increased sharply and oscillating magnifications are produced for  $d > 0.5$ . So the value of “d” needs to less than 0.5 to keep error magnification below unity.

## 5. Conclusion

In simplified explicit  $\theta$ -based solution of Richard’s equation, ratio of  $D_0 \Delta t / [\Delta x]^2$  affects the stability. At very low values, its error magnification is close to unity but less than 1 and at

values greater than 0.5, it shows too much error magnification. So, for best results the above ratio needs to lie in the range of 0.25 to 0.5.

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