On Non- Homogeneous Biquadratic Diophantine Equation $7(x^2+y^2) - 13xy = 31z^4$

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Abstract: Five different methods of the non-zero integral solutions of the homogeneous biquadratic Diophantine equation with five unknowns $7(x^2 + y^2) - 13xy = 31z^4$ are determined. Introducing the linear transformations $x = u + v, y = u - v, u \neq v \neq 0$ in $7(x^2 + y^2) - 13xy = 31z^4$, it reduces to $u^2 + 27v^2 = 31z^4$. We are solved the above equation through various choices and the different methods of solutions which are satisfied it. Some interesting relations among the special numbers and the solutions are exposed.

Keywords: Quadratic, non-homogenous, integer solutions, special numbers, polygonal, and pyramidal numbers

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Notations used

Tm,n : Polygonal number of rank n with sides m.
pn,m : Pyramidal number of rank m with side n
Gn : Gnomonic number of rank n

4,3rf : Fourth dimensional figurate number of rank r, whose generating polygon is a Triangle
4,4rf : Fourth dimensional figurate number of rank r, whose generating polygon is a Square
4,5rf : Fourth dimensional figurate number of rank r, whose generating polygon is a Pentagon
4,6rf : Fourth dimensional figurate number of rank r, whose generating polygon is a Hexagon
4,7rf : Fourth dimensional figurate number of rank r, whose generating polygon is a Heptagon
4,8rf : Fourth dimensional figurate number of rank r, whose generating polygon is a Octagon.

1. Introduction

The number theory is the queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [1-12]. In 2014, Jayakumar. P, Sangeetha. K, [12] have published a paper in finding the integer solutions of the homogeneous Biquaratic Diophantine equation $(x^3 - y^3)z = (W^2 - P^2)R^4$. In 2015, Jayakumar. P, Meena.J [14, 15] published two papers in finding integer solutions of the homogeneous Biquadratic Diophantine equation $(x^4 - y^4) = 26(z^2 - w^2)R^2$ and $(x^4 - y^4) = 40(z^2 - w^2)R^2$. Inspired by these, In this work, we are observed another interesting five different methods of the non-zero integral solutions of the non- homogeneous biquadratic Diophantine equation with three unknowns$7(x^2 + y^2) - 13xy = 31z^4$. Further, some elegant properties among the special numbers and the solutions are observed.

2. Description of Method

Consider the bi - quadratic Diophantine equation

$7(x^2+y^2) - 13xy = 31z^4$ (1)

We introduce the linear transformations

$x = u + v, y = u - v, u \neq v \neq 0$ (2)

Using (2) in (1), it gives to $u^2 + 27v^2 = 31z^4$ (3)
This gives us the non- zero different integer values to (1)

**Observations:-**
1. \( x(p,1)+8748 f_{4,p}^p - 12096 p^5 + 2241 T_{4,p} + G_{617p} = 0. \)
2. \( x(1,p) + 24 f_{4,3}^p + 220 p^5 - 283 T_{4,p} - G_{1560p} = 0 \text{(Mod2)} \)
3. \( y(1,p) + 72 f_{4,5}^p - 6 T_{4,p}^2 - 254 p^5 + 389 T_{4,p} + G_{1347p} = 0 \text{(Mod2)} \)
4. \( y(p,1) + 13122 f_{4,6}^p - 7722 p^5 - 999 T_{4,p} - G_{50p} + 2 = 0 \)
5. \( x(1, p) - y(1, p) - 48 f_{4,p}^p + 10 T_{4,p}^2 + 496 p^5 + 88 T_{4,p} + G_{2926p} = 0 \text{(Mod 2)} \)
6. \( x(p,1) + y(p,1) + 69984 f_{4,7}^p + 11664 T_{4,p}^2 - 8251 p^5 + 20196 T_{4,p} + G_{2926p} = 0 \text{(Mod 5)} \)
7. \( \frac{1}{3} z(2,0) \) is a perfect square.
8. \( \frac{1}{9} z(1,0) \) is a cubic integer.
9. \( z(1, 6) \) is a woodall number.
10. \( z(10,10) \) is a jacobsthal lucas number.

**2.2 Method: II**
In place of (4), let us take the form of ratio as

\[
\frac{u + 2z^2}{z^2 - v} = \frac{27(z^2 + v)}{u - 2z^2} = \frac{a}{b}, b \neq 0 \quad (8)
\]

The following techniques is similar as in the method - I, The relating integer values to (1) are found as

\[
x = x(p, q) = \frac{-58320p^3 - 80q^4 + 4332p^2q^3 + 2700pq^3 - 100p^3}{27p^2 + q^3}
\]
\[
y = y(p, q) = \frac{-20412p^2 - 28q^3 + 151p^2q^3 + 3132p^2q - 116pq^2}{27p^2 + q^3}
\]
\[
z = z(p, q) = \frac{-2p^2 + q^3}{27p^2 + q^3}
\]

**Observations:-**
1. \( x(p,1) + 349920 f_{4,6}^p - 355320 p^5 + 56688 T_{4,p} + G_{50p} = 0 \text{(Mod 3)} \)
2. \( x(1, p) + 1920 f_{4,5}^p - 160 T_{4,p}^2 - 1400 p^5 - 4352 T_{4,p} - G_{1432p} = 29 \text{(Mod 2011)} \)
3. \( y(1,p) + 672 f_{4,5}^p - 140 T_{4,p}^2 - 664 p^5 + 164 T_{4,p} - G_{1510p} = 31 \text{ (Mod 101)} \)
4. \( x(1, p) - y(1, p) + 1248 f_{4,7}^p - 208 T_{4,p}^2 - 1488 p^5 - 2436 T_{4,p} + G_{208p} = 7 \text{ (Mod 12634)} \)
5. \( y(p,1)+48988 f_{4,3}^p - 251208 p^5 - 100444 T_{4,p} - G_{6117p} = 0 \text{ (Mod 03)} \)
6. \( x(p,1)+y(p,1)+472392 f_{4,6}^p - 484056 p^5 + 78716 T_{4,p} + G_{108p} = 7 \text{ (Mod 17)} \)
7. \( \frac{1}{8} z(4,0) \) is a Nasty number.
8. \( z(1,3) \) is a perfect square.
9. \( z(5,6) \) is a cubic integer
10. \( \frac{1}{3} z(4,9) \) is a woodall number.
11. \( \frac{1}{2} z(1,9) \) is a Nasty number.

**2.3 Method: III**
Take 31 as 31 = \((2 + i\sqrt{27}) (2 - i\sqrt{27}) \)
Write \( z = z(a, b) = a^2 + 27b^2 \)

Using (9) and (10) is (3) and applying the factorization process, define \((u + i\sqrt{27})(a + i\sqrt{27})b^4 \) This give us \( u = 2a^4 + 1458b^4 - 324a^2b^2 - 108a^2b + 2916ab^3 \)
\( v = a^2 + 729b^2 - 162a^2b^2 + 8a^2b - 216ab^3 \)

Using (11) in (2), the relating integer values of (1) are furnished by
\[
x = x(a, b) = 3a^4 + 2187b^4 - 486a^2b^2 - 100ab^3 + 2700a^2b^2 - 162a^2b^2 - 108ab + 3132a^2b^3 \]
\( z = z(a, b) = a^2 + 27b^2 \)

**Observations:**
1. \( x(A, 1) - 72 f_{4,5}^d + 6 T_{4,a}^2 + 260 p_{4,d}^5 + 374 A_{4,d} = 613471 \)
   \( \text{= 0 (Mod 2)} \)
2. \( y(A, 1) - 12 f_{4,4}^d + 224 p_{4,d}^5 + 55 T_{4,a} - G_{1560a} = 0 \text{ (Mod 5)} \)
3. \( x(A, 1) - y(A, 1) - 48 f_{4,8}^d + 10 T_{4,a}^2 + 48 p_{4,d}^5 + 312 T_{4,a} + G_{212p} = 31 \text{ (Mod 47)} \)
4. \( x(A, 1) + y(A, 1) - 4 T_{4,a}^2 + 416 p_{4,d}^5 + 440 T_{4,a} + G_{216a} = 0 \text{ (Mod 5)} \)
5. \( x(A, 1) + y(A, 1) + z(A, 1) - 24 f_{4,6}^d + 440 p_{4,d}^5 + 435 T_{4,a} + G_{216a} = 0 \text{ (Mod 2)} \)
6. \( x(A, 1) + y(1, 1) = 0 \text{ (Mod 2)} \)
7. \( x(A, 1) - 52488 f_{4,4}^d + 8748 T_{4,a}^2 + 55836 p_{4,d}^5 - 12123 T_{4,a} - G_{21372a} = 0 \text{ (Mod 2)} \)
8. \( \frac{1}{7} (5,5) \) is a perfect square
9. \( \frac{1}{49} (0,7) \) is a cubic integer
10. \( \frac{1}{2} (0,4) \) is a Nasty number

**2.4 Method: IV**
In place of (9) take 31 as
\[
31 = \frac{(33 + i\sqrt{27})(33 - i\sqrt{27})}{36} \quad (12)
\]
The following techniques is same as in the method-III, the relating integer values of (1) are found as

\[
x = x(A, B) = 7344A^4 + 5353776B^4 - 1189728A^2B^2 - 139968AB^3 + 5184AB \]
\[
y = y(A, B) = 6912A^2 + 5038848B^4 - 1197744A^2B^2 - 51840A^2B + 1399680AB^3 \]
\[
z = z(A, B) = 16A^2 + 4563B^2 \]

**Observations:**
1. \( x(A, 1) - 88128 f_{4,4}^d + 48384 p_{4,d}^5 + 1202256 T_{4,a} \)
   \( + G_{7732a} = 0 \text{ (Mod 5)} \)
2. \( y(A, 1) - 165888 f_{4,4}^d + 185904 p_{4,d}^5 + 1102824 T_{4,a} \)
   \( - G_{679104a} = 11 \text{ (Mod 719834)} \)
3. \( x(A, 1) + y(A, 1) - 85536 f_{4,6}^4 + 178848 p_A^5 \)
   \(+ 2248560T_{4,A} - G_{629856A} = 0 \pmod{5} \)

4. \( x(1, A) - 321226584 f_{4,6}^4 + 321506520 p_A^5 \)
   \(- 52488004T_{4,A} - G_{2592A} = 0 \pmod{5} \)

5. \( y(1, A) - 120933252 f_{4,7}^4 + 10077696 T_{4,a}^2 + 97977600 \)
   \( p_A^5 - 2519424 T_{4,a} + G_{5064768A} = 3 \pmod{628} \)

6. \( x(1, A) - y(1, A) - 7558272 f_{4,7}^4 + 1259712 T_{4,a}^2 \)
   \(+ 11897280 p_A^5 - 3674160 T_{4,a} + G_{178848\times10^6} = 0 \pmod{5} \)

7. \( (0, 1) \) is a perfect square.

8. \( (1, 0) \) is a perfect square.

9. \( (A, 0) \) is a perfect square.

10. \( (1, A) \) is a perfect square.

2.5 Method V

Let us take (3) as \( u^2 + 27v^2 = (13 + i\sqrt{27}) (13 - i\sqrt{27}) \)

Using (9), (10) and (14) in (13) and applying the factorization process, define \( (u + i\sqrt{27} v) = (2 + i\sqrt{27}) (a + i\sqrt{27} b)^4 \)

\[ u = \frac{1}{14} [-27 + 27\sqrt{2} + 162a^2 + 162a^2b^2 + 1188980ab^3 - 1620a^3b] \quad (14) \]

\[ v = \frac{1}{14} [15a^3 + 10935b^2 - 2430a^2b^2 + 2916ab^3 - 4a^3b] \quad (15) \]

In sight of (2), the values of \( x, y \) are

\[ x = \frac{1}{14} [14a^4 + 10206b^2 - 2268a^2b^2 + 1183896ab^3 - 1624a^3b] \quad (16) \]

\[ y = \frac{1}{14} [-16a^4 - 1164b^2 - 2592a^2b^2 + 1178064ab^3 - 1616a^3b] \quad (17) \]

As our intention is to find integral solutions, taking \( a = 5A \) and \( b = 5B \) in (4), (16) and (17), the relating parametric integer values of (1) are found as

\[ x = x(A, B) = 625A^4 + 225625B^4 - 75200A^2B^2 + 997500AB^3 - 52500A^4B - 7500B^4 \]

\[ y = y(A, B) = -750A^2 - 270750B^4 + 85500A^2B + 94500AB^3 - 52000A^3B \]

\[ z = z(A, B) = 25A^2 + 475B^2 \]

Observations:

1. \( z(0, A) = 500 \) or \( z(0, B) = 25 \)

2. \( z(1, A) = 1 \) is a perfect square

3. \( x(1, 0) = 300 f_{4,7}^4 + 108410 p_A^5 - 17875T_{4,A} - G_{3228750A} = 0 \pmod{2} \)

Each of the following is a nasty number:

\[ z(1, 0) = \frac{3}{5}, z(1, 1) = \frac{6}{125}, z(0, 1), - \frac{1}{25}, y(1, 0) \]

3. Conclusion

In this work, we have observed various process of determining infinitely a lot of non-zero different integer values to the non-homogeneous bi-quadratic Diophantine equation \( 5(x^2 + y^2) - 9xy = 23z^4 \). One may try to find negative integer solutions of the above equations together with their similar observations.

References


\[ x^4 - y^4 = 26(z^2 - w^2) R^2 \]

\[ x^4 - y^4 = 40(z^2 - w^2) R^2 \]


[17] equation: \[ x^2 + y^2 - xy = 103z^3 \]


\[ x^4 - y^4 = 65(z^2 - w^2) R^2 \]

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