Six Series Equations involving Heat Polynomials

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Abstract: ***In this paper, it is shown that six series equations involving heat polynomials can be solved by reducing them to simultaneous Fredholm integral equations of second kind .These equations are not considered earlier by [3],[4].***

Keywords: Integral equation, Series equation, Fourier series, Integral theorems, Heat polynomials

1. Introduction

Dual, triple and quadruple series equations play an important role in finding the solution of mixed boundary value problems of elasticity, electrostatics and other fields of mathematical physics.Dual and triple equations involving orthogonal polynomials have been considered by many authors [1], [3], [4], [5]. Cooke [2] devised a method for finding the solution of quadruple series equations involving Fourier- Bessel series and obtained the solution using operator theory. In this paper, we have considered six series equations involving heat polynomials which are extensions of dual, triple and quadruple series equations considered by authors[1],[3],[4],[5].

2. Six Series Equations

We consider here the following sets of the six series equations of first kind:

Six series equations of the first kind are as follows:

$$\sum_{n=0}^{\infty}\frac{A_n}{\left(\mu+n+\frac{1}{2}+\rho\right)}P_{n+\rho,\sigma}(x,t)=\begin{cases}f_1(x,t) & ,0\le x<a\\ f_3(x,t) & ,b<x<c\\ f_5(x,t) & ,d<x<e\end{cases}\quad \textbf{(1)}$$

$$\sum_{n=0}^{\infty}\frac{t^{-n}l^{n}A_n}{\left(v+n+\frac{1}{2}+\rho\right)}P_{n+\rho,v}(x,-t)=\begin{cases}f_2(x,t) & ,a<x<b\\ f_4(x,t) & ,c<x<d\\ f_6(x,t) & ,e<x<\infty\end{cases}\quad \textbf{(2)}$$

where $f_i(x,t)$ are unknown functions for (i=1,2,3,4,5,6).$P_{n,\boldsymbol{v}}(x,-t)$ is a heat polynomial andCoefficients A_n to be determined.

3. Preliminary Results

In the course of analysis, we shall use the following results:
(i) The orthogonality relation for the heat polynomials

$$\int_0^{\infty}W_{m,\boldsymbol{v}}(x,t)P_{n,\boldsymbol{v}}(x,-t)d\boldsymbol{\Omega}(\boldsymbol{x})=\frac{\boldsymbol{\delta_{mn}}}{\boldsymbol{K_n}}\quad (3)$$

where δ_{mn} is the Kronecker delta,

$$d\Omega\,(x)=2^{\frac{1}{2}-\boldsymbol{v}}\left[\Gamma\left(v+\frac{1}{2}\right)\right]^{-1}x^{2v}dx\quad \textbf{(4)}$$

$$\text{and,}K_n=\frac{\Gamma\left[v+\frac{1}{2}\right]}{2^{4n}n!\,\Gamma\left[v+\frac{1}{2}+n\right]}\quad \textbf{(5)}$$

(ii) The series ,

$$S(x,\xi,t)=2^{\frac{1}{2}-\sigma}\sum_{n=0}^{\infty}\frac{\left(\frac{l}{t}\right)^{n}\Gamma\left(\mu+\frac{1}{2}+n+\rho\right)P_{n+\rho,v}(x,-t)W_{n+\rho,\sigma}(\xi,t)}{2^{4(n+\rho)}(n+\rho)!\;\Gamma\left(\sigma+\frac{1}{2}+n+\rho\right)\Gamma\left(v+\frac{1}{2}+n+\rho\right)}\quad \textbf{(6)}$$

$$S(x,\xi,t)=\frac{x^{1-2v}\xi^{1-2\sigma}e^{-\xi^2/4t}}{\Gamma m\Gamma(v-\sigma+m)}a_n{}^{*}\int_0^{\omega}\eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{v-\sigma+m-1}dy\quad \textbf{(7)}$$

$$\text{where,}a_n{}^{*}=\frac{l^{n}t^{-(m+n)}2^{2(1-m)}\Gamma\left(\mu+\frac{1}{2}+n+\rho\right)}{\Gamma\left(\sigma-m+\frac{1}{2}+n+\rho\right)},\quad (v-\sigma+m>0)\quad \textbf{(8)}$$

$\eta(y)=y^{2(\sigma-m)}e^{y^2/4t}$ and, $\omega=\min(\xi,x)$

If h(y) is strictly monotonically increasing and differentiable function in (a,b) andh(y) ≠0 in this interval, then the solutions to the Abel integral equations.

$$f(x)=\int_a^x\frac{\Phi(y)}{\{h(x)-h(y)\}^{\alpha}}dy\,,0<\alpha<1\quad \textbf{(9)}$$

and

$$f(x)=\int_x^b\frac{\Phi(y)}{\{h(y)-h(x)\}^{\alpha}}dy\,,0<\alpha<1\quad \textbf{(10)}$$

are given by ,

$$\Phi(y)=\frac{\sin(\alpha\pi)}{\pi}\frac{d}{dy}\int_a^y\frac{h'(x)F(x)}{\{h(y)-h(x)\}^{1-\alpha}}dx\quad \textbf{(11)}$$

and

$$\Phi(y)=-\frac{\sin(\alpha\pi)}{\pi}\frac{d}{dy}\int_y^b\frac{h'(x)F(x)}{\{h(x)-h(y)\}^{1-\alpha}}dx\quad \textbf{(12)}$$

respectively.

4. The Solution of Six Series Equations of the First Kind

Let us assume that,

$$\sum_{n=0}^{\infty}\frac{A_n}{\Gamma\left(\mu+\frac{1}{2}+n+\rho\right)}P_{n+\rho,\sigma}(x,-t)=\begin{cases}\emptyset_1(x,t) & ,a<x<b\\ \emptyset_2(x,t) & ,c<x<d\\ \emptyset_3(x,t) & ,e<x<\infty\end{cases}\quad \textbf{(13)}$$

where $\phi_1(x,t),\emptyset_2(x,t)$ and $\phi_3(x,t)$ are unknown functions.

Using orthogonality relation(3), we get A_n from equations (1),(2)

$$A_n = \frac{\Gamma\left(\sigma+\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}+n+\rho\right)}{2^{4(n+\rho)}(n+\rho)!}\left\{\int_0^a f_1(x,t) + \int_a^b \emptyset_1(x,t) + \int_b^c f_3(x,t) + \int_c^d \emptyset_2(x,t) + \int_d^e f_5(x,t) + \int_e^\infty \emptyset_3(x,t)\right\} W_{n+\rho,\sigma}(x,t)\,d\Omega(x) \quad \textbf{(14)}$$

Now substituting the value of A_n from (14) in equation (2), we get,

$$\int_0^a f_1(\xi,t)S(x,\xi,t)d\Omega(\xi) + \int_a^b \emptyset_1(\xi,t)S(x,\xi,t)d\Omega(\xi) + \int_b^c f_3(\xi,t)S(x,\xi,t)d\Omega + \int_c^d \emptyset_2(\xi,t)S(x,\xi,t)d\Omega(\xi) + \int_d^e f_5(\xi,t)S(x,\xi,t)d\Omega(\xi) + \int_e^\infty \emptyset_3(\xi,t)S(x,\xi,t)d\Omega(\xi) =$$

$$\frac{2^{\frac{1}{2}-\sigma}}{\Gamma(\sigma+\frac{1}{2})}\begin{cases} f_2(x,t) & ,a<x<b \\ f_4(x,t) & ,c<x<d \\ f_6(x,t) & ,e<x<\infty \end{cases} \quad \textbf{(15)}$$

Now starting with equation (5) , which can be written as

$$\int_a^x \emptyset_1(\xi,t)S_\xi(x,\xi,t)d\Omega(\xi) + \int_x^b \emptyset_1(\xi,t)S_x(x,\xi,t)d\Omega(\xi) = \frac{2^{\frac{1}{2}-\sigma}}{\Gamma\left(\sigma+\frac{1}{2}\right)} f_1(x,t)$$

$- \int_c^d \emptyset_2(\xi,t)S_x(x,\xi,t)d\Omega(\xi) - \int_e^\infty \emptyset_3(\xi,t)S_x(x,\xi,t)d\Omega(\xi)$, $a<x<b$ **(16)**

where,

$$F_1(x,t) = f_2(x,t) - \int_0^a \xi^{2\sigma} f_1(\xi,t)S(x,\xi,t)d(\xi) - \int_b^c \xi^{2\sigma} f_3(\xi,t)S(x,\xi,t)d(\xi) - \int_d^\infty \xi^{2\sigma} f_5(\xi,t)S(x,\xi,t)d(\xi) \quad \textbf{(17)}$$

with the help of equations (4), (7), we get

$$\int_a^x \emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi \int_0^\xi \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1}dy + \int_x^b \emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi \int_0^x \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1}dy$$

$$= \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a_n * x^{1-2\upsilon}}F_1(x,t) - \int_c^d \emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi \int_0^x \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1}dy - \int_e^\infty \emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi \int_0^x \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1}dy \quad \textbf{(18)}$$

Inverting the order of integration , we obtain $\int_0^a \eta(y)\,(x^2-y^2)^{\upsilon-\sigma+m-1}dy$.

$$\int_a^x \emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi + \int_a^x \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy \int_y^x \emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi + \int_0^x \eta(y)\,(x^2-y^2)^{\upsilon-\sigma+m-1}dy \int_x^b \emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi = \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a_n * x^{1-2\upsilon}}F_1(x,t)$$

$-\int_0^x \eta(y)(x^2 -$ y2υ−σ+m−1dycd∅2ξ,tξe−ξ24tξ2−y2m−1dξ_0xηyx2 −y2υ−σ+m−1dye∞∅3ξ,tξe−ξ24tξ2−y2m−1dξ $a<x<b$ **(19)**

$$\int_a^x \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy \int_y^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi$$
$$= \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a_n * x^{1-2\upsilon}}F_1(x,t)$$
$$-\int_0^a \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy \int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi$$
$$-\int_0^x \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy \int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi$$
$$-\int_0^x \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy \int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi \quad \textbf{(20)}$$

If we assume,

$$\int_y^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi = \overline{\emptyset}_1(y) \quad \textbf{(21)}$$

$$\int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi = \overline{\emptyset}_2(y) \quad \textbf{(22)}$$

$$\int_y^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi = \overline{\emptyset}_3(y) \quad \textbf{(23)}$$

Then equation **(20)** can be rewritten as below:

$$\int_a^x \frac{\eta(y)\overline{\emptyset}_1(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy = \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a_n * x^{1-2\upsilon}}F_1(x,t) -$$
$$\int_0^a \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy \int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi -$$
$$\int_0^x \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy \int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi -$$
$$\int_0^x \frac{\eta(y)}{(x^2-y^2)^{1-\upsilon+\sigma-m}}dy \int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi,$$
$$a < x < b \quad \textbf{(24)}$$

This is an Abel type integral equation and its solution, with the help of equation **(4)** is given by,

$$\eta(y)\overline{\emptyset}_1(y) = \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}\frac{d}{dy}\int_a^y \frac{2x}{(y^2-x^2)^{\upsilon+\sigma-m}}$$
$$\cdot\left[\frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a_n * x^{1-2\upsilon}}F_1(x,t)\right.$$
$$-\int_0^a \frac{\eta(z)}{(x^2-z^2)^{1-\upsilon+\sigma-m}}dz \int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi$$
$$-\int_o^x \frac{\eta(y)}{(x^2-z^2)^{1-\upsilon+\sigma-m}}dz \int_a^b \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi$$
$$\left.-\int_o^x \frac{\eta(y)}{(x^2-z^2)^{1-\upsilon+\sigma-m}}dz \int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right] dx \text{Or,}$$

$$\eta(y)\overline{\emptyset}_1(y)$$
$$= F'_1(y,t) - \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}\left[\int_0^a \eta(z)dz\frac{d}{dy}\int_a^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right.$$
$$+\int_0^x \eta(z)dz\frac{d}{dy}\int_a^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi$$
$$\left.+\int_0^x \eta(z)dz\frac{d}{dy}\int_a^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right]$$
$$\textbf{(25)}$$

where,

$$F'_1(y,t) = \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}\frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a_n *}\frac{d}{dy}\int_a^y \frac{2x^{2\upsilon}F_1(x,t)dx}{(y^2-x^2)^{\upsilon+\sigma-m}} \quad \textbf{(26)}$$

Inverting the order of integration in the II integral of R.H.S. in equation (24),we get,

$$\eta(y)\overline{\emptyset}_1(y) = F'_1(y,t) - \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}.\left[\int_0^a \eta(z)dz\frac{d}{dy}\int_a^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right.$$
$$+\int_o^a \eta(z)dz\frac{d}{dy}\int_a^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi$$
$$+\frac{d}{dy}\int_a^y \eta(z)dz\int_z^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi$$
$$+\int_o^a \eta(z)dz\frac{d}{dy}\int_a^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi$$
$$+\frac{d}{dy}\int_a^y \eta(z)dz\int_z^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}$$
$$\left.\cdot\int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right] \quad \textbf{(27)}$$

It can be easily proved that,

$$\frac{d}{dy}\int_a^y \frac{2xdx}{(y^2-x^2)^{\upsilon-\sigma+m}(x^2-z^2)^{1-\upsilon+\sigma-m}} = \frac{(a^2-z^2)^{\upsilon-\sigma+m}}{(y^2-z^2)(y^2-a^2)^{\upsilon-\sigma+m}} \quad \textbf{(28)}$$

And

$$\int_z^y \frac{2xdx}{(y^2-x^2)^{\upsilon-\sigma+m}(x^2-z^2)^{1-\upsilon+\sigma-m}} = \frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi} \quad \textbf{(29)}$$

Now using the results **(28)** and **(29)** in equation **(27)**, we obtain

$$\eta(y)\overline{\emptyset}_1(y) = F'_1(y,t)$$

$$-\frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}.\Bigg[\int_0^a \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+m}dz}{(y^2-z^2)\,(y^2-a^2)^{\upsilon-\sigma+m}}\int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi$$
$$+\int_0^a \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+m}dz}{(y^2-z^2)(y^2-a^2)^{\upsilon-\sigma+m}}\int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi$$
$$+\frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi}\frac{d}{dy}\int_a^y \eta(z)dz\int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi$$
$$+\int_0^a \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+m}dz}{(y^2-z^2)(y^2-a^2)^{\upsilon-\sigma+m}}\int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi$$
$$+\frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi}\frac{d}{dy}\int_a^y \eta(z)dz.\int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\Bigg] \quad \textbf{(30)}$$

Equations (21),(22) and (23) are also Abel type integral equations. Therefore the solution of these equations are given by

$$\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}} = -\frac{\sin(1-m)\pi}{\pi}\frac{d}{d\xi}\int_\xi^b \frac{2y\,\overline{\emptyset}_1(y)dy}{(y^2-\xi^2)^m} \quad \textbf{(31)}$$

$$\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}} = -\frac{\sin(1-m)\pi}{\pi}\frac{d}{d\xi}\int_\xi^d \frac{2y\,\overline{\emptyset}_2(y)dy}{(y^2-\xi^2)^m} \quad \textbf{(32)}$$

$$\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}} = -\frac{\sin(1-m)\pi}{\pi}\frac{d}{d\xi}\int_\xi^d \frac{2y\,\overline{\emptyset}_3(y)dy}{(y^2-\xi^2)^m} \quad \textbf{(33)}$$

Therefore,

$$\int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi = -\frac{\sin(1-m)\pi}{\pi(a^2-z^2)^{-m}}\int_a^b \frac{2x\overline{\emptyset}_1(x)dx}{(x^2-a^2)^m(x^2-z^2)} \quad \textbf{(34)}$$

$$\int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi = -\frac{\sin(1-m)\pi}{\pi(c^2-z^2)^{-m}}\int_c^d \frac{2x\overline{\emptyset}_2(x)dx}{(x^2-c^2)^m(x^2-z^2)} \quad \textbf{(35)}$$

$$\int_c^d \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi = -\frac{\sin(1-m)\pi}{\pi(e^2-z^2)^{-m}}\int_c^d \frac{2x\overline{\emptyset}_3(x)dx}{(x^2-e^2)^m(x^2-z^2)} \quad \textbf{(36)}$$

Substituting the values from equations (34),(35)and (36) in equation (30), we obtain

$$\eta(y)\overline{\emptyset}_1(y) = F'_1(y,t) -$$

$$\frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}\frac{\sin(1-m)\pi}{\pi}.\Bigg[\int_0^a \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+2m}dz}{(y^2-z^2)\,(y^2-a^2)^{\upsilon-\sigma+m}}\int_a^b \frac{2x\overline{\emptyset}_1(x)dx}{(x^2-a^2)^m(x^2-z^2)}$$
$$+\int_0^a \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+m}dz}{(c^2-z^2)^{-m}(y^2-z^2)\,(y^2-a^2)^{\upsilon-\sigma+m}}\int_c^d \frac{2x\overline{\emptyset}_2(x)dx}{(x^2-c^2)^m(x^2-z^2)}$$
$$+\frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi}.\frac{d}{dy}\int_a^y \frac{\eta(z)dz}{(c^2-z^2)^{-m}}\int_c^d \frac{2x\overline{\emptyset}_2(x)dx}{(x^2-c^2)^m(x^2-z^2)}$$
$$+\int_0^a \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+m}dz}{(e^2-z^2)^{-m}(y^2-z^2)\,(y^2-a)^{\upsilon-\sigma+m}}\int_e^\infty \frac{2x\overline{\emptyset}_3(x)dx}{(x^2-e^2)^m(x^2-z^2)}$$
$$+\frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi}.\frac{d}{dy}\int_a^y \frac{\eta(z)dz}{(e^2-z^2)^{-m}}.\int_e^\infty \frac{2x\overline{\emptyset}_3(x)dx}{(x^2-e^2)^m(x^2-z^2)}\Bigg]$$

With the help of equations(31),(32) and(33) , the above equation takes the form

$$\eta(y)\overline{\emptyset}_1(y) = F'_1(y,t) - \int_a^b \overline{\emptyset}_1(x)P_1(x,y)\,dx - \int_c^d \overline{\emptyset}_2(x)P_2(x,y)\,dx - \int_e^\infty \overline{\emptyset}_3(x)P_3(x,y)\,dx,$$

$a < x < b$ **(37)**

where,

$$P_1(x,y) = \frac{\sin(1-\upsilon+\sigma-m)\pi\,\sin(1-m)\pi}{\pi^2(y^2-a^2)^{\upsilon-\sigma+m}}\frac{2x}{(x^2-a^2)^m}\int_0^a \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+2m}}{(y^2-z^2)(x^2-z^2)}dz \quad \textbf{(38)}$$

$$P_2(x,y) = \frac{\sin(1-\upsilon+\sigma-m)\,\pi\sin(1-m)\pi}{\pi^2(y^2-a^2)^{\upsilon-\sigma+m}}\frac{2x}{(x^2-c^2)^m}$$

$$\int_0^a \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+m}(c^2-z^2)^m}{(y^2-z^2)(x^2-z^2)}dz + \frac{\sin(1-m)\pi}{\pi}\frac{2x}{(x^2-c^2)^m}\frac{d}{dy}. \int_a^y \frac{\eta(z)(c^2-z^2)^m}{(x^2-z^2)}dz \quad (39)$$

$$P_3(x,y) = \frac{\sin(1-\nu+\sigma-m)\pi\,\sin(1-m)\pi}{\pi^2(y^2-a^2)^{\upsilon-\sigma+m}}\frac{2x}{(x^2-e^2)^m}$$

$$\int_0^a \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+m}(e^2-z^2)^m}{(y^2-z^2)(x^2-z^2)}dz + \frac{\sin(1-m)\pi}{\pi}\frac{2x}{(x^2-e)^m}\frac{d}{dy}\int_a^y \frac{\eta(z)(e^2-z^2)^m}{(x^2-z^2)}dz \quad (40)$$

Now starting with equation

$$(5)\int_a^b \emptyset_1(\xi,t)S_\xi(x,\xi,t)d\Omega(\xi) + \int_c^x \emptyset_2(\xi,t)S_\xi(x,\xi,t)d\Omega(\xi) + \int_x^d \emptyset_2(\xi,t)S_\xi(x,\xi,t)d\Omega(\xi) + \int_e^\infty \emptyset_3(\xi,t)S_\xi(x,\xi,t)d\Omega(\xi) = \frac{2^{\frac{1}{2}-\sigma}}{\Gamma\left(\sigma+\frac{1}{2}\right)}F_2(x,t) \quad ,c<x<d \quad (41)$$

where,

$$F_2(x,t) = f_4(x,t) - \int_0^a \xi^{2\sigma}f_1(\xi,t)S(x,\xi,t)d\xi - \int_b^c \xi^{2\sigma}f_3(\xi,t)S(x,\xi,t)d\xi - \int_d^\infty \xi^{2\sigma}f_5(\xi,t)S(x,\xi,t)d\xi \quad (42)$$

With the help of equations (4) and (7),we get

$$\int_c^x \emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi .\int_0^\xi \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1}dy$$

$$+\int_x^d \emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi .\int_0^x \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1}dy$$

$$= \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a*x^{1-2\nu}}F_2(x,t) - \int_a^b \emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi .\int_0^x \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1}dy$$

$$-\int_e^\infty \emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi \int_0^x \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1}dy \quad (43)$$

Now inverting the order of integration, we obtain,

$$\int_0^c \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy .\int_c^x \emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi$$

$$+\int_c^x \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy .\int_y^x \emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi$$

$$+\int_0^x \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy .\int_x^d \emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi$$

$$= \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a*x^{1-2\nu}}F_2(x,t) - \int_0^x \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy .\int_a^b \emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi$$

$$-\int_0^x \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy .\int_e^\infty \emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi \quad (44)$$

Or,

$$\int_c^x \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}}\int_y^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi = \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a*x^{1-2\nu}}F_2(x,t)$$

$$-\int_0^c \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}}\int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi - \int_0^x \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}}\int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi$$

$$-\int_0^x \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}}\int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi \quad (45)$$

Hence,

$$\int_c^x \frac{\eta(y)\bar{\phi}_2(y)dy}{(x^2-y^2)^{1-v+\sigma-m}} = \frac{\Gamma m\Gamma(v-\sigma+m)}{a*x^{1-2v}}F_2(x,t) - \int_0^c \frac{\eta(y)dy}{(x^2-y^2)^{1-v+\sigma-m}}\int_c^d \frac{\phi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi - \int_0^x \frac{\eta(y)dy}{(x^2-y^2)^{1-v+\sigma-m}}\int_a^b \frac{\phi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi - \int_0^x \frac{\eta(y)dy}{(x^2-y^2)^{1-v+\sigma-m}}\int_e^\infty \frac{\phi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}}d\xi \quad ,c<x<d \tag{46}$$

This is an Abel type integral equation and its solution ,with the help of equation (4) is given by

$$\eta(y)\bar{\phi}_2(y) = \frac{\sin(1-v+\sigma-m)\pi}{\pi}\frac{d}{dy}\int_c^y \frac{2x}{(y^2-x^2)^{v-\sigma+m}}\cdot\left[\frac{\Gamma m\Gamma(v-\sigma+m)}{a*x^{1-2v}}F_2(x,t) - \int_0^c \frac{\eta(z)dz}{(x^2-z^2)^{1-v+\sigma-m}}\cdot\int_c^d \frac{\phi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi - \int_0^x \frac{\eta(z)dz}{(x^2-z^2)^{1-v+\sigma-m}}\int_a^b \frac{\phi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi - \int_0^x \frac{\eta(z)dz}{(x^2-z^2)^{1-v+\sigma-m}}\int_e^\infty \frac{\phi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right]dx$$

Or,

$$\eta(y)\bar{\phi}_2(y) = F'_2(y,t) - \frac{\sin(1-v+\sigma-m)\pi}{\pi}\left[\int_0^c \eta(z)dz\frac{d}{dy}\int_c^y \frac{2xdx}{(y^2-x^2)^{v-\sigma+m}(x^2-z^2)^{1-v-\sigma+m}}\cdot\int_c^d \frac{\phi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \int_0^x \eta(z)dz.\frac{d}{dy}\int_c^y \frac{2xdx}{(y^2-x^2)^{v-\sigma+m}(x^2-z^2)^{1-v-\sigma+m}}\int_a^b \frac{\phi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \int_0^x \eta(z)dz.\frac{d}{dy}\int_c^y \frac{2xdx}{(y^2-x^2)^{v-\sigma+m}(x^2-z^2)^{1-v-\sigma+m}}\int_e^\infty \frac{\phi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right] \tag{47}$$

$$\text{where,} F'_2(y,t) = \frac{\sin(1-v+\sigma-m)\pi}{\pi}\cdot\frac{\Gamma m\Gamma(v-\sigma+m)}{a*}\cdot\frac{d}{dy}\int_c^y \frac{2x^{2v}F_2(x,t)}{(y^2-x^2)^{v-\sigma+m}}dx \tag{48}$$

Inverting the order of integration in second integral of R.H.S. in equation **(46)**

$$\eta(y)\bar{\phi}_2(y) = F'_2(y,t) - \frac{\sin(1-v+\sigma-m)\pi}{\pi}\left[\int_0^c \eta(z)dz\frac{d}{dy}\int_c^y \frac{2xdx}{(y^2-x^2)^{v+\sigma-m}(x^2-z^2)^{1-v+\sigma-m}}\cdot\int_c^d \frac{\phi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \int_o^c \eta(z)dz\frac{d}{dy}.\int_c^y \frac{2xdx}{(y^2-x^2)^{v+\sigma-m}(x^2-z^2)^{1-v+\sigma-m}}\int_a^b \frac{\phi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \frac{d}{dy}\int_c^y \eta(z)dz\int_z^y \frac{2xdx}{(y^2-x^2)^{v+\sigma-m}(x^2-z^2)^{1-v+\sigma-m}}\int_a^b \frac{\phi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \int_o^c \eta(z)dz\frac{d}{dy}\int_c^y \frac{2xdx}{(y^2-x^2)^{v+\sigma-m}(x^2-z^2)^{1-v+\sigma-m}}\int_e^\infty \frac{\phi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \frac{d}{dy}\int_c^y \eta(z)dz\int_z^y \frac{2xdx}{(y^2-x^2)^{v+\sigma-m}(x^2-z^2)^{1-v+\sigma-m}}\int_e^\infty \frac{\phi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right] \tag{49}$$

It can be easily proved that,

$$\frac{d}{dy}\int_c^y \frac{2xdx}{(y^2-x^2)^{v-\sigma+m}(x^2-z^2)^{1-v+\sigma-m}} = \frac{(c^2-z^2)^{v-\sigma+m}}{(y^2-z^2)(y^2-c^2)^{v-\sigma+m}} \tag{50}$$

And

$$\int_z^y \frac{2xdx}{(y^2-x^2)^{v-\sigma+m}(x^2-z^2)^{1-v+\sigma-m}} = \frac{\pi}{\sin(1-v+\sigma-m)\pi} \tag{51}$$

Now using the results **(50)** and **(51)** in equation **(49),**we obtain

$$\eta(y)\bar{\phi}_2(y) = F'_2(y,t) - \frac{\sin(1-v+\sigma-m)\pi}{\pi}\cdot\left[\int_0^c \frac{\eta(z)(c^2-z^2)^{v-\sigma+m}dz}{(y^2-z^2)(y^2-c^2)^{v-\sigma+m}}\int_a^b \frac{\phi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \frac{\pi}{\sin(1-v+\sigma-m)\pi}\frac{d}{dy}\int_c^y \eta(z)dz\int_a^b \frac{\phi_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \int_0^c \frac{\eta(z)(c^2-z^2)^{v-\sigma+m}dz}{(y^2-z^2)(y^2-c^2)^{v-\sigma+m}}\int_c^d \frac{\phi_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \int_0^c \frac{\eta(z)(c^2-z^2)^{v-\sigma+m}dz}{(y^2-z^2)(y^2-c^2)^{v-\sigma+m}}\int_e^\infty \frac{\phi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi + \frac{\pi}{\sin(1-v+\sigma-m)\pi}\frac{d}{dy}\int_c^y \eta(z)dz.\int_e^\infty \frac{\phi_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}}d\xi\right] \tag{52}$$

Substituting the values from equations (34),(35)and (36) in equation (52), we obtain

$$\eta(y)\bar{\phi}_2(y) = F'_2(y,t) - \frac{\sin(1-v+\sigma-m)\pi}{\pi}\frac{\sin(1-m)\pi}{\pi}\cdot\left[\int_0^a \frac{\eta(z)(a^2-z^2)^{v-\sigma+2m}dz}{(y^2-z^2)(y^2-a^2)^{v-\sigma+m}}\int_a^b \frac{2x\bar{\phi}_1(x)dx}{(x^2-a^2)^m(x^2-z^2)}\right.$$

$$+\int_0^a \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+m}dz}{(c^2-z^2)^{-m}(y^2-z^2)\,(y^2-a^2)^{\upsilon-\sigma+m}}\int_c^d \frac{2x\overline{\emptyset}_2\,(x)dx}{(x^2-c^2)^m(x^2-z^2)}$$

$$+\frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi}\cdot\frac{d}{dy}\int_a^y \frac{\eta(z)dz}{(c^2-z^2)^{-m}}\int_c^d \frac{2x\overline{\emptyset}_2\,(x)dx}{(x^2-c^2)^m(x^2-z^2)}$$

$$+\int_0^a \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+m}dz}{(e^2-z^2)^{-m}(y^2-z^2)\,(y^2-a)^{\upsilon-\sigma+m}}\int_e^\infty \frac{2x\overline{\emptyset}_3\,(x)dx}{(x^2-e^2)^m(x^2-z^2)}$$

$$+\frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi}\cdot\frac{d}{dy}\int_a^y \frac{\eta(z)dz}{(e^2-z^2)^{-m}}\cdot\int_e^\infty \frac{2x\overline{\emptyset}_3\,(x)dx}{(x^2-e^2)^m(x^2-z^2)}\Bigg]$$

with the help of equations(31),(32) and(33) , the above equation takes the form

$$\eta(y)\overline{\emptyset}_2\,(y) = F'_2(y,t)-\int_a^b \overline{\emptyset}_1\,(x)Q_1(x,y)\,dx - \int_c^d \overline{\emptyset}_2\,(x)Q_2(x,y)\,dx - \int_e^\infty \overline{\emptyset}_3\,(x)Q_3(x,y)\,dx\,,\qquad c<x<d \qquad \textbf{(53)}$$

where,

$$Q_1(x,y)=\frac{\sin(1-\nu+\sigma-m)\,\pi\sin(1-m)\pi}{\pi^2(y^2-c^2)^{\upsilon-\sigma+m}}\frac{2x}{(x^2-a^2)^m}\int_0^C \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+2m}}{(y^2-z^2)(x^2-z^2)}dz + \frac{\sin(1-m)\pi}{\pi}\frac{2x}{(x^2-a^2)^m}\frac{d}{dy}\int_C^y \frac{\eta(z)(a^2-z^2)^m}{(x^2-z^2)}dz \qquad \textbf{(54)}$$

$$Q_2(x,y)=\frac{\sin(1-\nu+\sigma-m)\,\pi\sin(1-m)\pi}{\pi^2(y^2-c^2)^{\upsilon-\sigma+m}}\frac{2x}{(x^2-c^2)^m}\int_0^C \frac{\eta(z)(c^2-z^2)^{\upsilon-\sigma+2m}}{(y^2-z^2)(x^2-z^2)}dz \qquad \textbf{(55)}$$

$$Q_3(x,y)=\frac{\sin(1-\nu+\sigma-m)\,\pi\sin(1-m)\pi}{\pi^2(y^2-a^2)^{\upsilon-\sigma+m}}\frac{2x}{(x^2-e^2)^m}\int_0^C \frac{\eta(z)(a^2-z^2)^{\upsilon-\sigma+m}(e^2-z^2)^m}{(y^2-z^2)(x^2-z^2)}dz + \frac{\sin(1-m)\pi}{\pi}\frac{2x}{(x^2-e)^m}\frac{d}{dy}\int_c^y \frac{\eta(z)(e^2-z^2)^m}{(x^2-z^2)}dz \qquad \textbf{(56)}$$

Now starting with equation (5)

$$\int_a^b \emptyset_1\,(\xi,t)S_\xi(x,\xi,t)d\Omega(\xi) + \int_c^d \emptyset_2\,(\xi,t)S_\xi(x,\xi,t)d\Omega(\xi) + \int_e^x \emptyset_3\,(\xi,t)S_\xi(x,\xi,t)d\Omega(\xi) + \int_x^\infty \emptyset_3\,(\xi,t)S_\xi(x,\xi,t)d\Omega(\xi) = \frac{2^{\frac{1}{2}-\sigma}}{\Gamma\left(\sigma+\frac{1}{2}\right)}F_4(x,t)\qquad ,e<x<\infty \qquad \textbf{(57)}$$

where,

$$F_4\,(x,t) = f_6(x,t)-\int_0^a \xi^{2\sigma}f_1\,(\xi,t)S(x,\xi,t)d\xi - \int_b^c \xi^{2\sigma}f_3\,(\xi,t)S(x,\xi,t)d\xi - \int_d^\infty \xi^{2\sigma}f_5\,(\xi,t)S(x,\xi,t)\,d\xi \qquad \textbf{(58)}$$

with the help of equations (4) and (7),we get

$$\int_e^x \emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi \int_0^\xi \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1}dy + \int_x^\infty \emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi \int_0^x \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1}dy$$

$$=\frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a*x^{1-2\nu}}F_4\,(x,t)-\int_a^b \emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi\int_0^x \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1} - \int_c^d \emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}d\xi \int_0^x \eta(y)(\xi^2-y^2)^{m-1}(x^2-y^2)^{\upsilon-\sigma+m-1} \qquad \textbf{(59)}$$

Now inverting the order of integration, we obtain,

$$\int_0^e \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy\int_e^x \emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi + \int_e^x \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy\int_y^x \emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi + \int_0^x \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy\int_x^\infty \emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi$$

$$=\frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a*x^{1-2\nu}}F_4\,(x,t) - \int_0^x \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy\int_a^b \emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi - \int_0^x \eta(y)(x^2-y^2)^{\upsilon-\sigma+m-1}dy\int_c^d \emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}(\xi^2-y^2)^{m-1}d\xi \qquad \textbf{(60)}$$

Or,

$$\int_e^x \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} \int_y^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}} d\xi = \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a * x^{1-2\upsilon}} F_4(x,t) - \int_0^e \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} \int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}} d\xi - \int_0^x \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} \int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}} d\xi - \int_0^x \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} \int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}} d\xi \quad (61)$$

Hence,

$$\int_e^x \frac{\eta(y)\overline{\emptyset}_3(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} = \frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a * x^{1-2\upsilon}} F_4(x,t) - \int_0^e \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} \int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}} d\xi - \int_0^x \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} \int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}} d\xi - \int_0^x \frac{\eta(y)dy}{(x^2-y^2)^{1-\upsilon+\sigma-m}} \int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-y^2)^{1-m}} d\xi \quad , e < x < \infty$$

(62)

This is an Abel type integral equation and its solution ,with the help of equation (4) is given by

$$\eta(y)\overline{\emptyset}_3(y) = F'_4(y,t)$$

$$-\frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}\Bigg[\int_0^x \eta(z)dz\,.\frac{d}{dy}\int_e^y \frac{2xdx}{(y^2-x^2)^{\upsilon-\sigma+m}(x^2-z^2)^{1-\upsilon-\sigma+m}}\,.\int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi + \int_0^x \eta(z)dz\,.\frac{d}{dy}\int_e^y \frac{2xdx}{(y^2-x^2)^{\upsilon-\sigma+m}(x^2-z^2)^{1-\upsilon-\sigma+m}}\,.$$

$$\int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi + \int_0^e \eta(z)dz\,.\frac{d}{dy}\int_c^y \frac{2xdx}{(y^2-x^2)^{\upsilon-\sigma+m}(x^2-z^2)^{1-\upsilon-\sigma+m}}\,.\int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi\Bigg]$$ where,

(63)

$$F'_4(y,t) = \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}\,.\frac{\Gamma m\Gamma(\upsilon-\sigma+m)}{a *}\frac{d}{dy}\int_e^y \frac{2x^{2\upsilon}F_4(x,t)}{(y^2-x^2)^{\upsilon-\sigma+m}} dx$$

Inverting the order of integration in second integral of R.H.S. in equation **(62)**

$$\eta(y)\overline{\emptyset}_3(y) = F'_4(y,t) - \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}$$

$$\Bigg[\int_0^e \eta(z)dz\frac{d}{dy}\int_e^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}$$

$$.\int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi + \frac{d}{dy}\int_e^y \eta(z)dz$$

$$\int_z^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi$$

$$+\int_o^e \eta(z)dz\frac{d}{dy}\int_e^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}$$

$$+\int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi + \int_o^e \eta(z)dz\frac{d}{dy}$$

$$\int_e^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi$$

$$+\frac{d}{dy}\int_e^y \eta(z)dz\int_z^y \frac{2xdx}{(y^2-x^2)^{\upsilon+\sigma-m}(x^2-z^2)^{1-\upsilon+\sigma-m}}\int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi\Bigg]$$

(64)

It can be easily proved that,

$$\frac{d}{dy}\int_e^y \frac{2xdx}{(y^2-x^2)^{\upsilon-\sigma+m}(x^2-z^2)^{1-\upsilon+\sigma-m}} = \frac{(e^2-z^2)^{\upsilon-\sigma+m}}{(y^2-z^2)(y^2-e^2)^{\upsilon-\sigma+m}} \quad \mathbf{(65)}$$

And

$$\int_z^y \frac{2xdx}{(y^2-x^2)^{\upsilon-\sigma+m}(x^2-z^2)^{1-\upsilon+\sigma-m}} = \frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi} \quad \mathbf{(66)}$$

Now using the results **(65)** and **(66)** in equation **(64),**we obtain

$$\eta(y)\overline{\emptyset}_3(y) = F'_4(y,t) - \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi} \cdot \left[\int_0^e \frac{\eta(z)(e^2-z^2)^{\upsilon-\sigma+m}dz}{(y^2-z^2)\,(y^2-e^2)^{\upsilon-\sigma+m}} \int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi \right.$$
$$+ \frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi}\frac{d}{dy}\int_e^y \eta(z)dz \int_a^b \frac{\emptyset_1(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi$$
$$+\int_0^e \frac{\eta(z)(e^2-z^2)^{\upsilon-\sigma+m}dz}{(y^2-z^2)(y^2-e^2)^{\upsilon-\sigma+m}} \int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi$$
$$+\int_0^e \frac{\eta(z)(e^2-z^2)^{\upsilon-\sigma+m}dz}{(y^2-z^2)(y^2-e^2)^{\upsilon-\sigma+m}} \int_e^\infty \frac{\emptyset_3(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi$$
$$\left. + \frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi}\frac{d}{dy}\int_e^y \eta(z)dz \,. \int_c^d \frac{\emptyset_2(\xi,t)\xi e^{-\frac{\xi^2}{4t}}}{(\xi^2-z^2)^{1-m}} d\xi \right] \quad \mathbf{(67)}$$

Substituting the values from equations (34),(35)and (36) in equation (67), we obtain

$$\eta(y)\overline{\emptyset}_3(y) = F'_4(y,t) - \frac{\sin(1-\upsilon+\sigma-m)\pi}{\pi}\frac{\sin(1-m)\pi}{\pi} \cdot \left[\int_0^e \frac{\eta(z)(e^2-z^2)^{\upsilon-\sigma+2m}dz}{(a^2-z^2)^{-m}(y^2-z^2)\,(y^2-e^2)^{\upsilon-\sigma+m}} \int_a^b \frac{2x\overline{\emptyset}_1(x)dx}{(x^2-a^2)^m(x^2-z^2)} \right.$$
$$+\frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi}\cdot\frac{d}{dy}\int_e^y \frac{\eta(z)dz}{(a^2-z^2)^{-m}} \int_a^b \frac{2x\overline{\emptyset}_1(x)dx}{(x^2-a^2)^m(x^2-z^2)}$$
$$+\int_0^e \frac{\eta(z)(e^2-z^2)^{\upsilon-\sigma+m}dz}{(c^2-z^2)^{-m}(y^2-z^2)\,(y^2-e^2)^{\upsilon-\sigma+m}} \int_c^d \frac{2x\overline{\emptyset}_2(x)dx}{(x^2-c^2)^m(x^2-z^2)}$$
$$+\frac{\pi}{\sin(1-\upsilon+\sigma-m)\pi}\cdot\frac{d}{dy}\int_e^y \frac{\eta(z)dz}{(c^2-z^2)^{-m}} \int_c^d \frac{2x\overline{\emptyset}_2(x)dx}{(x^2-c^2)^m(x^2-z^2)}$$
$$\left.+\int_0^e \frac{\eta(z)(e^2-z^2)^{\upsilon-\sigma+m}dz}{(e^2-z^2)^{-m}(y^2-z^2)\,(y^2-e^2)^{\upsilon-\sigma+m}} \int_e^\infty \frac{2x\overline{\emptyset}_3(x)dx}{(x^2-e^2)^m(x^2-z^2)}\right] \quad \mathbf{(68)}$$

with the help of equations(31),(32) and(33) , the above equation takes the form

$$\eta(y)\overline{\emptyset}_3(y) = F'_4(y,t) - \int_a^b \overline{\emptyset}_1(x)R_1(x,y)\,dx - \int_c^d \overline{\emptyset}_2(x)R_2(x,y)\,dx - \int_e^\infty \overline{\emptyset}_3(x)R_3(x,y)\,dx\,,$$
$$e < x < \infty \quad \mathbf{(69)}$$

where,

$$R_1(x,y) = \frac{\sin(1-\nu+\sigma-m)\,\pi\,\sin(1-m)\pi}{\pi^2(y^2-e^2)^{\upsilon-\sigma+m}}\frac{2x}{(x^2-a^2)^m}\int_0^e \frac{\eta(z)(a^2-z^2)^m(e^2-z^2)^{\upsilon-\sigma+m}}{(y^2-z^2)(x^2-z^2)}dz + \frac{\sin(1-m)\pi}{\pi}\frac{2x}{(x^2-a^2)^m}\frac{d}{dy}\int_e^y \frac{\eta(z)(a^2-z^2)^m}{(x^2-z^2)}dz \quad \mathbf{(70)}$$

$$R_2(x,y) = \frac{\sin(1-\nu+\sigma-m)\,\pi\,\sin(1-m)\pi}{\pi^2(y^2-e^2)^{\upsilon-\sigma+m}}\frac{2x}{(x^2-c^2)^m}\int_0^e \frac{\eta(z)(c^2-z^2)^m(e^2-z^2)^{\upsilon-\sigma+2m}}{(y^2-z^2)(x^2-z^2)}dz + \frac{\sin(1-m)\pi}{\pi}\frac{2x}{(x^2-c^2)^m}\frac{d}{dy}\int_e^y \frac{\eta(z)(c^2-z^2)^m}{(x^2-z^2)}dz \quad \mathbf{(71)}$$

$$R_3(x,y) = \frac{\sin(1-\nu+\sigma-m)\,\pi\,\sin(1-m)\pi}{\pi^2(y^2-e^2)^{\upsilon-\sigma+m}}\frac{2x}{(x^2-e^2)^m}\int_0^e \frac{\eta(z)(e^2-z^2)^{\upsilon-\sigma+2m}}{(y^2-z^2)(x^2-z^2)}dz \quad \mathbf{(72)}$$

Equations (27),(53) and (69) are simultaneous Fredholm integral equations of the second kind. With the help of these equations we can calculate the values of $\overline{\emptyset}_1(y)$, $\overline{\emptyset}_2(y)$ and $\overline{\emptyset}_3(y)$. Then using equations (31),(32) and (33),we can calculate the values of $\emptyset_1(\xi,t)$, $\emptyset_2(\xi,t)$ and $\emptyset_3(\xi,t)$ .After these calculations A_n can be determined by equation (14).

5. Particular Cases

When e → ∞ in equations (1) and (2)then above equations reduce to the five series equations and the solutions obtained for the six series reduce to the solution of five series equations. Similarly if d → ∞ and c → d in equations (1) and (2) the series reduces to quadruple series equations and the solution obtained here agrees with[4]and the solutions of dual and triple series can be obtained as a particular case studied earlier[1]and [3].

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