Lattice Points on the Homogeneous Cone $x^2+y^2=26z^2$

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Abstract: ***The ternary quadratic homogeneous equation representing cone $x^2+y^2= 26z^2$ is considered and analyzed for finding its non zero distinct integral solutions. Five different patterns of integer points satisfying the cone under consideration are obtained. A few interesting relation between the solutions and special number patterns are presented.***

Keywords: Homogeneous cone, Ternary Quadratic, Integral solutions, special numbers.

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Notations Used

$T_{m,n}$ - Polygonal number of rank n with size m - P_n^r- Pyramidal number of rank n with size m - g -Gnomonic number of rank n- Pr_n - Pronic number of rank n - $Ct_{16,n}$- Centered hexadecagonal pyramidal number of rank n- OH_n - Octahedral number of rank n- SO_n - Stella octangular number of rank n - ky_n - kynea number - $carl_n$ -carol number- w_n - woodall number- S_n-Star number- C_n-Cullen number.

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. This communication concerns with another interesting ternary quadratic equation representing $x^2 + y^2= 26z^2$ a cone for determining its infinitely many non-zero integral points. Also a few interesting properties among the solutions are presented.

2. Method of Analysis

The ternary quadratic equation to be solved for its non-zero integer solution is

$$x^2 + y^2 = 26z^2 \quad (1)$$

Assume $z(a, b) = a^2 + b^2$, where $a, b > 0$ (2)

We illustrate below five different patterns of non-zero distinct integer solutions to (1)

2.1 Pattern: I

Write $26 = (5 + i)(5 – i)$ (3)

Substituting (2) & (3) in (1), and applying the method of factorization, define

$$(x + iy)(x – iy) = (5 + i)(5 – i)(a + ib)^2 (a – ib)^2$$

Equating real and imaginary parts, we get

$$x = x(a, b) = 5a^2 – 5b^2 – 2ab \quad (4)$$
$$y = y(a, b) = a^2 – b^2 + 10ab \quad (5)$$

Properties

1. $x(A,A) + 2T_{4,A} = 0$
2. $y(A, A + 1) – Ct_{16,A} – 2T_{4,A} \equiv 0 \pmod 2$
3. $z[A(A + 1), A(A + 1)] – 2(P_A)^2 = 0$
4. $x[1, A(2A^2 –1)] –5[1, A(2A^2 –1)] + 52SO_A = 0$
5. $y(1, 1) – PT_3 \equiv 0 \pmod 5$
6. $x[(2A^2 +1), A] –5y[(2A^2 + 1),A] +156OH_A = 0$
7. $y(A, 2A^2 +1) + z(A, 2A^2 +1) –2T_{4,A}- 30OH_A = 0$
8. $x(2A^2 –1, A) –3y(2A^2 –1, A) –2z(2A^2 –1, A) +7T_{4,A}+32SO_A= 0$
9. $x[(A + 1),(A + 2)] + CH_A – 2T_{23,A} + 20T_{4,A} \equiv 0 \pmod 2$
10. 3x (A, A) is a nasty number.

2.2 Pattern: II

Instead of (3), Write 26 as

$$26 = (1 + 5i)(1 – 5i) \quad (6)$$

Following the procedure presented as in pattern : 1, the corresponding values of x and y are

$$x = x(a, b) = a^2 – b^2 – 10ab \quad (7)$$
$$y = y(a, b) = 5a^2 – 5b^2 +2ab \quad (8)$$

Properties:-

1. $5x(A, A) – y(A, A) + 52T_{4,A} = 0$
2. $y(1, 1) –P_2^5 \equiv 0 \pmod 2$
3. $y[A,(A + 1)(2A + 1)] – x[A, (A + 1)(2A + 1)] –312P_a^4 = 0$
4. $y(1,1) – x(1,1) – OH_3$ = Carol number.
5. $y(A,1) – 5x(A,1) – 5z(A,1) +5T_{4,A} – G_{6A} +P_2^5 \equiv 0 \pmod 2$
6. $y(2A^2 –1, A) –5x(2A^2 –1, A) –52SO_A= 0$
7. $x(A^2,A+1) –PT_A+22T_{4,A}+ 24OH_A$= carol number
8. $x(A,A) + 8y(A, A)$ is a nasty number
9. $x(1, 2A + 1) + Ct_{16,A}–S_A + 20T_{3,A} – 8T_{4,A}= 0$
10. 6x (A, A) is a nasty number

2.3 Pattern: III

Equation (1) is written in the form of ratio as

$$\frac{x+z}{5z+y}=\frac{5z-y}{x-z}=\frac{A}{B},\quad B\neq 0,$$

which is equivalent to the system of equations

$$B(x+z)-A(5z+y)=0 \quad (9)$$
$$B(5z-y)-A(x-z)=0 \quad (10)$$

Applying the method of cross multiplication the integer solutions of (1) are given by

$$x(a,b)=A^2-B^2+10AB \quad (11)$$
$$y(a,b)=A^2-5B^2-AB \quad (12)$$
$$z(a,b)=A^2+B^2 \quad (13)$$

Properties:

1. $x[1,A(2A^2-1)]+z[1,A(2A^2-1)]-10SO_A-P_1^5=0$
2. $y(2A,2A)-5x(2A,2A)+220T_{4,A}=0$
3. $x(A,1)+10(A,1)-11T_{4,A}+O_4+Ky_1=0$
4. $z(1,1)-W_2\equiv 0 \pmod 5$
5. $x(2A,1)-y(2A,1)-20T_{4,A}-G_{11A}=0$
6. $y(A,A+1)+z(A,A+1)-6T_{4,A}+2T_{3,A}=0$
7. $y[A(A+1),1]-z[A(A+1),1]-22T_{3,A}\equiv 0 \pmod 2$
8. $y(2A-1,A)-x(2A-1,A)+4T_{4,A}+11T_{6,A}=0$
9. $3x(A,A)+2y(A,A)+2z(A,A)$ is a Nasty number.
10. $3z(A,A)$ is a Nasty number.

2.4 Pattern: IV

Equation (1) is written as

$$26z^2-y^2=x^2*1 \quad (14)$$

Assume $x(a,b)=26a^2-b^2)$ (15)

Write (1) as $1=(\sqrt{26}+5)(\sqrt{26}-5)$ (16)

Substituting (15) & (16) in (14) and applying the method of factorization, define

$$(\sqrt{26}\,z+y)=(\sqrt{26}\,a+b)^2(\sqrt{26}-5)$$

Equating the rational and irrational factors we get

$x=x(A,B)=26a^2-b^2$

$$y=y(A,B)=130a^2+5b^2-52ab \quad (17)$$
$$z=z(A,B)=26a^2+b^2-10ab \quad (18)$$

Properties

1. $y(A,A)-5z(A,A)+102\,T_{4,A}=0$
2. $z[1,(A(2A^2-1)]+10SO_A=0$
3. $z(A,A+1)-27T_{4,A}-G_A+10P_A\equiv 0 \pmod 2$
4. $y[(2A^2-1),1]-5[(2A^2-1),1]+52(G_A)^2-P_3^5\equiv 0 \pmod 2$
5. $y(1,A)-5T_{4,A}+G_{26A}-P_6^5\equiv 0 \pmod 5$
6. $5x(2A^2+1,A)-y(2A^2+1)+156OH_A\equiv 0 \pmod 2$
7. $x(2A,2A)$ is a Perfect square
8. $\frac{1}{42}[y(A,A)-5x(A,A)]$ is a Cubic integer
9. $y(1,1)-5x(1,1)+z(1,1)+C_3=0$
10. $6x(1,1)+z(1,1)-Carl_4\equiv 0 \pmod 2$

2.5 Pattern: V

Assume $x(a,b)=26a^2-b^2)$ (19)

Substituting (19) in (1) and applying the method of factorization, define

$(\sqrt{26}\,a+b)^2=(\sqrt{26}\,z+y)$

Equating rational and irrational, we get

$$x(a,b)=(26a^2-b^2) \quad (20)$$
$$y(a,b)=26a^2+b^2 \quad (21)$$
$$z=z(a,b)=2ab \quad (22)$$

Properties:

1. $z[A(A+1),A(A+1)]-2(P_A)^2=0$
2. $x[(A+1),(A+2)]-25T_{4,A}-G_{24A}+P_3^6\equiv 0 \pmod 2$
3. $y(A,A+1)-27T_{4,A}-G_A\equiv 0 \pmod 2$
4. $z(A,2A^2-1)-2SO_A=0$
5. $x[A,(A+1)(2A+1)]+52T_{4,A}-P_A^4=0$
6. $z(A,2A-1)-2T_{6,A}=0$
7. $z(A,8A-7)-2T_{18,A}=0$
8. $12z(1,1)$ is a Nasty number
9. $x(A,A)-y(A,A)+2T_{4,A}=0$
10. $z(A,A+1)-4T_{3,A}=0$

3. Conclusion

In this work, the ternary quadratic Diophantine equations referring a conies are analysed for its non-zero distinct integral points. A few interesting properties between the solutions and special numbers are presented. To conclude, one may search for other patterns of solutions and their corresponding properties for the cone under consideration.

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