On The Homogeneous Biquadratic Equation With 5 Unknowns: x^4-y^4=65 (z^2-w^2) R^2

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Abstract: ***The Homogenous biquadratic equation with five unknowns given by x^4-y^4=65 (z^2-w^2) R^2 is considered and analyzed for finding its non zero distinct integral solutions. Introducing the linear transformations x = u + v, y = u – v, z =2uv +1, w =2uv-1 and employing the method of factorization different patterns of non zero distinct integer solutions of the equation under the above equation are obtained. A few interesting relations between the integral solutions and the special numbers namely Polygonal numbers, Pyramidal numbers, Centered Polygonal numbers, Centered Pyramidal numbers, Thabit-ibn-Kurrah number, Star number, Carol number, woodall number, kynea number, pentatope number,stellaoctangul number,octahedral number, Mersenne number are exhibited.***

Keywords: Homogeneous equation, Integral solutions, Polygonal numbers, Pyramidal and special number.

2010 Mathematics Subject Classification: 11D09

Notations used:

$T_{m,n}$ - Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

g_n - Gnomonic number of rank n

Pr_n - Pronic number of rank n

$Ct_{16,n}$- Centered hexadecagonal pyramidal number of rank n

OH_n - Octahedral number of rank n

SO_n - Stella octangular number of rank n

\ky_n - kynea number

$carl_n$ -carol number

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-12]. In this context one may refer [4-10] for various problems on the biquadratic Diophantine equations. However, often we come across non-homogeneous biquadratic equations and as such one may require its integral solution in its most general form. This paper concern with the homogeneous biquadratic equation with five unknowns $x^4–y^4=65(z^2–w^2)R^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

2. Method of Analysis

The biquadratic equation with five unknowns to be solved for its non-zero distinct integral solution is

$$x^4 - y^4 = 65\,(z^2 - w^2)\,R^2 \qquad (1)$$

Consider the transformations

$$x = u + v,\ y = u – v,\ z = 2uv + 1,\ w = 2uv – 1 \qquad (2)$$

On substituting (2) in (1), we get

$$u^2 + v^2 = 65R^2 \qquad (3)$$

2.1 Pattern: I

$$\text{Assume } 65 = (8 + i)\,(8 – i) \qquad (4)$$

$$\text{and } R= a^2 + b^2 = (a + i\,b)\,(a – i\,b) \qquad (5)$$

Using (4) and (5) in (3) and employing the method of factorization we get.

$(u + i\,v)\,(u – iv) = (8 + i)\,(8 – i)\,(a + i\,b)^2\,(a – i\,b)^2$

On equating the positive and negative factors, we have,

$(u + i\,v) = (8 + i)\,(a + i\,b)^2$

$(u + i\,v) = (8 – i)\,(a – i\,b)^2$

On equating real and imaginary parts, we get

$u = u\,(a, b) = 8a^2 – 8b^2 – 2ab$

$v = v\,(a, b) = a^2 – b^2 + 16ab$

On substituting u and v in (2) we get the values of x, y, z and w. The non-zero distinct integrals values of x, y, z, w and R satisfying (1) are given by

$x = x\,(a, b) = 9a^2 – 9b^2 + 14ab$

$y = y\,(a, b) = 7a^2 – 7b^2 – 18ab$

$z = z\,(a, b) = 2(8a^4 + 8b^4 – 48a^2 b^2 + 126a^3b – 126ab^3) +1$

$w = w\,(a, b) = 2(8a^4 +8b^4 – 48a^2 b^2 + 126a^3b –126ab^3) –1$

$R = R\,(a, b) = a^2 + b^2$

Properties:-

1. $x\,(2, a) + y\,(2, a) + Ct_{16,a} + 8t_{4,a} – W_4 = 0$
2. $R\,(a+1, a+1) – 2t_{4,A} – G_{2a} \equiv 0 \pmod 3$

3. $z(1,b) - W(1,b) - OH_2 \equiv 0 \pmod 4$
4. $x[a(a+1),1] - 9y[a(a+1),1] - 260P_a = 0$
5. $7x[a(2a^2-1),1] - 9y[a(2a^2-1),1] - 260S_a = 0$
6. $R(3,3) - P_3^5 = 0$
7. $x(2,2) - y(2,2) \equiv 0 \pmod 2$
8. $7x[1,a(2a^2+1)] - 9y[1,a(2a^2+1)] - 864OH_a = 0$
9. $z(1,1) + w(1,1) + T_{23,4} \equiv 0 \pmod 2$
10. $7x[1,a(a+1)] - 9y[1,a(a+1)] - 260(P_a)^2 = 0.$

2.2 Pattern: II

Also 65 can be written in equation (3) as

$$65 = (1+8i)(1-8i) \quad (6)$$

Using (5) and (6) in equation (3) it is written in factorizable form as

$(u+iv)(u-iv) = (1+8i)(1-8i)(a+ib)^2(a-ib)^2$

On equating the positive and negative factors, we get,

$(u+iv) = (1+8i)(a+ib)^2$
$(u-iv) = (1-8i)(a-ib)^2$

On equating real and imaginary parts, we have

$u = u(a,b) = a^2 - b^2 - 16ab$
$v = v(a,b) = 8a^2 - 8b^2 + 2ab$

Substituting the values of u and v in (2), the non-zero distinct values of x, y, z, w and R satisfying (1) are given by

$x = x(a,b) = 9a^2 - 9b^2 - 14ab$
$y = y(a,b) = -7a^2 + 7b^2 - 18ab$
$z = z(a,b) = 2(8a^4 + 8b^4 - 48a^2b^2 - 126a^3b + 126ab^3) + 1$
$w = w(a,b) = 2(8a^4 + 8b^4 - 48a^2b^2 - 126a^3b + 126ab^3) - 1$
$R = R(a,b) = a^2 + b^2$

Properties:-

1. $7x[(2a-1)^2,1] + 9y[(2a-1)^2,1] + 260(G_a)^2 = 0$
2. $y(2a,1) + 7R(2a,1) + G_{9a} - W_1 \equiv 0 \pmod 2$
3. $R(2a,2a) - 8t_{4,a} = 0$
4. $x(a,a+1) + y(a,a+1) + Ct_{16,a} + 24t_{4,a} + G_{14a} = 0$
5. $x(a,1) + R(a,1) - 10T_{4,a} + G_{7a} + TK_2 \equiv 0 \pmod 2$
6. $z(2,2) + w(2,2) + T_{10,23} = Carol\,number.$
7. $7x[a(a+1),1] + 9y[a(a+1),1] + 288P_a = 0.$
8. $7x[1,(2a-1)^2] + 9y[1,(2a-1)^2] + 260(G_a)^2 = 0$
9. $9y[1,a(2a^2+1)] - 7x[1,a(2a^2+1)] - 378(OH_a)^2 - 192OH_a + T_{10,6} = 0$
10. $7x(1,1) + 9y(1,1) + CS_6 = woodall\,number.$

2.3 Pattern: III

Rewrite (3) as

$$1 * u^2 = 65R^2 - v^2 \quad (7)$$

$$\text{Assume } u = 65a^2 - b^2 = (\sqrt{65} + a)(\sqrt{65} - b) \quad (8)$$

$$\text{Write 1 a } 1 = (\sqrt{65}+8)(\sqrt{65}-8) \quad (9)$$

Using (8) and (9) in (7) it is written in factorizable form as

$(\sqrt{65}+8)(\sqrt{65}-8)(\sqrt{65}\,a+b)^2(\sqrt{65}\,a-b)^2$
$= (\sqrt{65}\,R+v)(\sqrt{65}\,R-v)$

On equating the rational and irrational parts, we get

$(\sqrt{65}+8)(\sqrt{65}\,a+b)^2 = (\sqrt{65}\,R+v)$
$(\sqrt{65}-8)(\sqrt{65}\,a-b)^2 = (\sqrt{65}\,R-v)$

On equating the real and imaginary parts, we get

$R = R(a,b) = 65a^2 + b^2 + 16ab$
$v = v(a,b) = 520a^2 + 8b^2 + 130ab$

Substituting the values of u and v in (2), the non – zero distinct integral values of x, y, z, R and w satisfying (1) are given by

$x = x(a,b) = 585a^2 + 7b^2 + 130ab$
$y = y(a,b) = -455a^2 - 9b^2 - 130ab$
$z = z(a,b) = 2(33800a^4 - 8b^4 - 130ab^3 + 8450a^3b) + 1$
$w = w(a,b) = 2(33800a^4 - 8b^4 - 130ab^3 + 8450a^3b) - 1$
$R = R(a,b) = 65a^2 + b^2 + 16ab$

Properties:

1. $x[A, A(2A^2-1)] + y[A, A(2A^2-1)] - 30T_{4,A} + 2(SO_A)^2 = 0$
2. $R(A, 2A-1) - 101T_{4,A} + G_{10A} = 0$
3. $R(2A, 2A) - 328T_{4,A} = 0$
4. $x[1, A(A+1)] - 7R[1, A(A+1)] - 18P_A - PT_6 \equiv 0 \pmod 4$
5. $y(A,1) + 9R(A,1) - 130T_{4,A} - G_{7A} - PT_1 = 0$
6. $x[a(2a^2+1),1] + y[a(2a^2+1),1] - 390(OH_a)^2 = 0$
7. $x(1,1) + y(1,1) \equiv 0 \pmod 2$
8. $x[a, a(2a^2+1)] - 585T_{4,a} - 21(OH_a)^2 - 390OH_a = 0$
9. $z(1,1) + w(1,1) \equiv 47 \pmod{3584}$
10. $x[a(a+1),1] - 130Pa - 585(Pa)^2 - P_2^6 = 0.$

2.4 Pattern: IV Rewrite (3) as

$$1 * v^2 = 65R^2 - u^2 \quad (11)$$

$$\text{Write 1 as } 1 = \frac{(\sqrt{65}+1)(\sqrt{65}-1)}{64} \quad (12)$$

$$\text{Assume } v = 65a^2 - b^2 = (\sqrt{65}\,a - b)(\sqrt{65}\,a + b) \quad (13)$$

Using (12) and (13) in (11), it is written in factorizable form as

$$\frac{(\sqrt{65}+1)(\sqrt{65}-1)}{64}(\sqrt{65}\,a-b)^2(\sqrt{65}\,a+b)^2$$
$$= (\sqrt{65}\,R-u)(\sqrt{65}\,R+u) \quad (14)$$

On equating the rational and irrational factors we get,

$$\left.\begin{aligned} R &= R(a,b) = \tfrac{1}{8}(65a^2+b^2+2ab) \\ u &= u(a,b) = \tfrac{1}{8}(65a^2+b^2+130ab) \end{aligned}\right\} (15)$$

Replacing „a" by 8A and „b" by 8B in the above equations (13) and (15), we get

$R = R(A,B) = 520A^2 + 8B^2 + 16AB$
$u = u(A,B) = 520A^2 + 8B^2 + 1040AB$
$v = v(A,B) = 4160A^2 - 64B^2$

On substituting the values of u and v in (2), the non –zero distinct integrals values of x, y, z, w and R satisfying (1) are given by

$x = x(A,B) = 4680A^2 - 56B^2 + 1040AB$
$y = y(A,B) = -3640A^2 + 72B^2 + 1040AB$

$z = z(A, B) = 2(2163200A^4 - 512B^4 + 4326400A^3B - 66560AB^3) + 1$
$w = w(A, B) = 2(2163200A^4 - 512B^4 + 4326400A^3B - 66560AB^3) - 1$
$R = R(A, B) = 520A^2 + 8B^2 + 16AB$

Properties:-

1. $x(A+1, 1) - y(A+1, 1 - 8320\,T_{4,} - G_{8320A} - S_5 \equiv (\text{mod } 2)$
2. $y(1A) - 9R(1,A) - G_{448A} \equiv 0 \pmod 3$
3. $R[A(2A^2-1), 1] - 520(SO_A)^2 + 16(SO_A) - W_2 =$ Star number
4. $z(A, 1) - W(A,1) \equiv 0 \pmod 2$
5. $x(1,B) + 7R(1,B) - P^5_{25} - G_{516B} \equiv 0 \pmod 2$
6. $R(1,1) - 4$ is a Nasty number.
7. $x(1,1) - y(1,1) - P^5_{25} - Gno_{34} = 0$
8. $72x[a,(2a^2+1)] + 56y[a,(2a^2+1)] - 133120T_{4,a} - 49920OH_a = 0$
9. $8x(1, 2a+1) + 224P_a - G_{1040a} - J_8 =$ Cullen number.
10. $z(1,1) + w(1,1) \equiv 32 \pmod{802816}$

2.5 Pattern: V

Write (3) as $u^2 - R^2 = 64R^2 - v^2$

$$(u + R)(u - R) = (8R + v)(8R - v), \quad (16)$$

which is expressed in the form of ratio as

$$\frac{u+R}{8R+v} = \frac{8R-v}{u-R} = \frac{A}{B},\ B \neq 0 \quad (17)$$

This is equivalent to the following two equations
$-uA + R(8B + A) - VB = 0$
$uB + R(B - 8A) - VA = 0$

On solving the above equations by the method of cross multiplication we get,
$u = u(A, B) = -A^2 - B^2 - 16AB$
$R = R(A, B) = -A^2 - B^2$
$v = v(A, B) = 8A^2 - 8B^2 - 2AB$

Substituting the values of u and v in (2), the non – zero distinct integral values of x, y, z, w and R satisfying (1) are given by,
$x = x(A, B) = 7A^2 - 9B^2 - 18AB$
$y = y(A, B) = -9A^2 + 7B^2 - 14AB$
$z = z(A, B) = 2[-8A^4 + 8B^4 + 32A^2B^2 - 126A^3B + 126AB^3] + 1$
$w = w(A, B) = 2[-8A^4 + 8B^4 + 32A^2B^2\ 126A^3B + 126AB^3] - 1$
$R = R(A, B) = -A^2 - B^2$

Properties :

1. $9x[1, A(A+1)] + 7y[1, A(A+1)] + 32(P_A)^2 - 260T_{4,A} - G_{130A} =$ woodall number
2. $R(A+1, 1) + T_{4,A} + G_A \equiv 0 \pmod 3$
3. $y[1, A(2A^2-1)] + 7x[1, A(2A^2-1)] + 14\,SO_A + W_3 - Ky_1 = 0$
4. $R(2A, 2A) + 8t_{4,A} = 0$
5. $x(1,1) + 7R(1,1) + P^5_4 =$ Nasty number
6. $x[a(2a^2+1), 1] - 21(OH_a)^2 + 18OH_a$ is a cubic integer
7. $w(1,1) + z(1,1) + R(1,1) - PT_6 = 0$
8. $y(1, a+1) - 7T_{4,a}$ is a perfect square
9. $R[a(2a^2+1), 1] - 3(OH_a)^2 =$ carol number.
10. $9x[(a+1),(a+2)] + 7y[(a+1),(a+2)] + 292_{T_{4,a}} - G_{454a} \equiv 0 \pmod 2$.

3. Conclusion

It is worth to note that in (2), the transformations for z and w may be considered as z = 2u + v and w = 2u –v. For this case, the values of x, y and R are the same as above where as the values of z and w changes for every pattern. To conclude one may consider biquadratic equations with multivariables ($\geq$5) and search for their non-zero distinct integer solutions along with their corresponding properties.

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