Asymptotic Behavior of Solution of a Periodic Mutualistic System

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Abstract: ***We focus on the system of reaction-diffusion equations. We prove the existence of steady state solution of mutualistic system with constant coefficients. And our purpose is to estimates for periodic solutions of periodic system. We derive the asymptotic behavior of periodic system.***

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1. Introduction

In this paper, we consider the system of reaction diffusion equations [2, 3]. The equations is given by the following system:

$$\begin{cases} u_t = \Delta u + u[a(t,x) - b(t,x)u + c(t,x)v], \\ v_t = \Delta v + v[d(t,x) + e(t,x)u - f(t,x)v]. \end{cases} \tag{1.1}$$

here a, b, c, d, e and f are sufficiently smooth functions defined on a cylinder $\Omega \times [0,T]$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, Δ denotes the Laplacian with respect to the variables $x = (x_1, x_2, \dots, x_n) \in \Omega$, $\partial/_{\partial v}$ denotes derivative in the direction of the outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and $u(t,x), v(t,x)$ is a solution of (1.1).
We assume that $a, \dots, f$ are strictly positive and periodic in the time variable t with period $T > 0$.
The boundary condition is supposed by

$$\left.\frac{\partial u}{\partial v}\right|_{\partial\Omega\times[0,T]} = \left.\frac{\partial v}{\partial v}\right|_{\partial\Omega\times[0,T]} = 0 \tag{1.2}$$

and the initial condition is given by

$$u(x,t)|_{t=0} = u0(x), v(x,t)|_{t=0} = v0(x). \tag{1.3}$$

2. The Existence of Steady State Solutions

Consider the following steady state problem:

$$\begin{cases} u_t = \Delta u + u[a - bu + cv], \\ v_t = \Delta v + v[d + eu - fv]. \end{cases} \tag{2.1}$$

here a, b, c, d, e and f are positive constants and $u(t,x), v(t,x)$ is a solution of (2.1).

Steady state solution satisfies the following equations:

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$$f_1(u,v) = u(a - bu + cv) = 0,$$
$$f_2(u,v) = v(d + eu - fv) = 0$$

and then we compute the Jacobian

$$A = \begin{bmatrix} (a - bu + cv) - bu & cu \\ ev & (d + eu - fv) - fv \end{bmatrix}$$

Now we find intersection points:

$$L_1 = a - bu + cv = 0, \tag{2,2}$$
$$L_2 = d + eu - fv = 0. \tag{2,3}$$

from Eq (2.2) in u axis the point is $(\frac{a}{b}, 0)$, from Eq (2.3) in v axis the point is $(0, \frac{d}{f})$ now we solving the simultaneous equations (2.2) − (2.3) and then we find $(u^\star, v^\star) = \left(\frac{af+cd}{bf-ce}, \frac{ae+bd}{bf-ce}\right)$ as $\frac{b}{e} > \frac{c}{f}$ there are four equilibriums points (0, 0), $(\frac{a}{b}, 0)$, $\left(0, \frac{d}{f}\right)$ and $(u^\star, v^\star)$ now we are discuss the stability for these points:

(i) $det(A - I\lambda)_{(0,0)} = det\begin{bmatrix} a - \lambda & 0 \\ 0 & d - \lambda \end{bmatrix} = 0$ then $\lambda_1, \lambda_2 > 0$ then this point is unstable.

(ii) $det(A - I\lambda)_{(\frac{a}{b},0)} = det\begin{bmatrix} a - 2b.\frac{a}{b} - \lambda & \frac{ca}{b} \\ 0 & d + \frac{ea}{b} - \lambda \end{bmatrix} = 0$ then $\lambda_1 < 0, \lambda_2 > 0$ the point is unstable.

(iii) $det(A - I\lambda)_{(0,\frac{d}{f})} = det\begin{bmatrix} a + \frac{cd}{f} - \lambda & 0 \\ \frac{ed}{f} & -d - \lambda \end{bmatrix} = 0$ then $\lambda_1 > 0, \lambda_2 < 0$
the point is unstable.

(iv) $det(A - I\lambda)_{(u^\star,v^\star)} = det\begin{bmatrix} -bu^\star - \lambda & cu^\star \\ ev^\star & -fv^\star - \lambda \end{bmatrix} = 0$

$$= (-bu^\star - \lambda)(-fv^\star - \lambda) - ceu^\star v^\star = 0$$
$$= \lambda^2 + (bu^\star + fv^\star)\lambda + (bf - ce)u^\star v^\star = 0$$
$$\lambda_{1,2} = \frac{-(bu^\star + fv^\star) \pm \sqrt{(bu^\star + fv^\star)^2 - 4(bf - ce)u^\star v^\star}}{2}$$

$Re\lambda_{1,2} < 0$ if $\frac{b}{c} > \frac{e}{f}$ under this condition the point $(u^\star, v^\star)$ is stable.

Lemma 1 Let $P = (u_1, v_1), Q = (u_2, v_2)$ be any two distinct points in R_i and let Γ_i be any smooth curve lying in R_i with end points P, Q where R_i is any one of the four regions as following:

$R_1 : f_1(p,q) < 0, f_2(p,q) < 0, R_2 : f_1(p,q) < 0, f_2(p,q) > 0$

$R_3 : f_1(p,q) > 0, f_2(p,q) < 0, R_4 : f_1(p,q) > 0, f_2(p,q) > 0$

These four regions are closely related to the following corresponding differential Inequalities.

$\Gamma_1 : p' \geq f_1(p,q), q' \geq f_2(p,q), \Gamma_2 : p' \geq f_1(p,q), q' \leq f_2(p,q)$

$\Gamma_3 : p' \leq f_1(p,q), q' \geq f_2(p,q), \Gamma_4 : p' \leq f_1(p,q), q' \leq f_2(p,q)$

then there exists a pair of smooth functions $(p(t), q(t))$ with values on Γ_i for all $t \geq 0$ such that:

(1) $(p(0), q(0)) = (u_1, v_1)$; $\lim_{t\mapsto\infty}(p(t), q(t)) = (u_2, v_2)$

(2) (p, q) satisfies the corresponding differential inequalities in Γ_i

Proof: we show the lemma for Γ_{2i} in R_2 since it is representative and is more relevant to later applications. Consider the case where Γ_2 is the straight line $\overline{PQ}$. Since P, Q are in R_2 there exists $\delta > 0$ such that:

$$\begin{cases} \max\{f_1(u,v), (u,v) \in \overline{PQ} \leq -\delta\}, \\ \min\{f_2(u,v), (u,v) \in \overline{PQ} \geq \delta\} \end{cases} \quad (2.4)$$

Define:

$$\begin{cases} p(t) = u_2 + (u_1 - u_2)e^{-\varepsilon t}, (t \geq 0) \\ p(t) = v_2 + (v_1 - v_2)e^{-\varepsilon t}, (t \geq 0) \end{cases} \quad (2.5)$$

where $\varepsilon > 0$ is a constant to be chosen. Then (p, q) lies on Γ_2 for all t and satisfies property (1) choose $\varepsilon \leq \min\{\delta|u_1 - u2-1, \delta v1-v2-1$. Then

$p'(t) = -\varepsilon(u_1 - u_2)e^{-\varepsilon t} \geq -\delta$; $q'(t) = -\varepsilon(v_1 - v_2)e^{-\varepsilon t} \leq \delta$.

It follows from (2.4) that property (2) holds. Next consider Γ_2 as an arbitrary smooth curve PQ with length S. Then Γ_2 may be represented by the parametric equation.

$$p = p_1(s), q = q_1(s), 0 \leq s \leq S.$$

With

$$(p(0), q(0)) = (u_1, v_1), (p(S), q(S)) = (u_2, v_2).$$

Define:

$$\begin{cases} p(t) = p_1(S(1 - e^{-\varepsilon t})), (t \geq 0) \\ q(t) = q_1(S(1 - e^{-\varepsilon t})), (t \geq 0) \end{cases} \quad (2.6)$$

For some $\varepsilon > 0$. Then (p, q) lies on Γ_2 and satisfies property (1). Since Γ_2 is smooth there exists a constant k such that:

$$p'(t) = p_1'(S(1 - e^{-\varepsilon t}))S\varepsilon e^{-\varepsilon t} \leq -\epsilon k S$$
$$q'(t) = q_1'(S(1 - e^{-\varepsilon t}))S\varepsilon e^{-\varepsilon t} \leq \epsilon k S.$$

It follows by choosing $\varepsilon < (kS)^{-1}\delta$ that property (2) from (2.4) this proves the lemma.

Theorem 2.1.

Suppose that $\frac{b}{c} > \frac{e}{f}$ *holds then for* $u_0(x) > 0$,

$$u_0(x) \not\equiv 0 \text{ and } v_0(x) > 0, v_0(x) \not\equiv 0, x \in \Omega$$
$$\lim_{t\to\infty}(u(x,t), v(x,t)) = (u^*, v^*).$$

Proof: Let L_1, L_2 be the straight lines in the $(u,v) - plane$ given by (2.2) – (2.3) we choose a point $\overline{P} = (\overline{u}_1, \overline{v}_1)$ in R_1 with $\overline{u}_1 > u_0(x), \overline{v}_1 > v_0(x)$ and a point $P = (u_1, v_1)$ in R_4 with $u_1 < u_0(x), v_1 < v_0(x)$ for any given $\varepsilon > 0$ then we choose $Q = (u_2, v_2)$ in R_4 and $\overline{Q} = (\overline{u}_2, \overline{v}_2)$ in R_1 such that:

$u_2 > u^* - \varepsilon; v_2 > v^* - \varepsilon$ and $\overline{u}_2 < u^* + \varepsilon; \overline{v}_2 < v^* + \varepsilon$.

That $i.e$ $u_1 \leq u_0(x) \leq \overline{u}_1; v_1 \leq v_0(x) \leq \overline{v}_1$.

Let Γ_1 and Γ_4 be the respective line in R_1, R_4 with the end points $\overline{P}, \overline{Q}$ and $, Q$. Then by the Lemma(1) there exists a functions $(p(t), q(t)), (\overline{p}(t), \overline{q}(t))$ with $(p(0), q(0)) = (u_1, v_1), (\overline{p}(0), \overline{q}(0)) = (\overline{u}_1, \overline{v}_1)$

such that (p, q) satisfies the inequalities

$p(t) \leq p(a - bp + cq), q(t) \leq q(d + ep - fq)$

and $(\overline{p}, \overline{q})$ satisfies the inequalities

$\overline{p}'(t) \geq \overline{p}(a - b\overline{p} + c\overline{q}), \overline{q}'(t) \geq \overline{q}(d + e\overline{p} + f\overline{q})$

Since

$p(0) \leq u_0(x) \leq \overline{p}(0), q(0) \leq v_0(x) \leq \overline{q}(0)$ and $p(t) \leq u(t,x) \leq \overline{p}(t), q(t) \leq v(t,x) \leq \overline{q}(t)$.

But by property (1) of the lemma(1)

$p(t) \to u_2 > u^* - \varepsilon; q(t) \to v_2 > v^* - \varepsilon$ and $\overline{p}(t) \to \overline{u}_2 < u^* + \varepsilon; \overline{q}(t) \to \overline{v}_2 < v^* + \varepsilon$ as $t \to \infty$.

We conclude by letting $t \to \infty$ in $p(t) \leq u(t,x) \leq \overline{p}(t), q(t) \leq v(t,x) \leq \overline{q}(t)$ and the arbitrariness of ε that:

$u(t,x) \to u^*, v(t,x) \to v^*$ as $t \to \infty$ then

$$\lim_{t\to\infty}(u(t,x), v(t,x)) = (u^*, v^*).$$

3. Estimates for Periodic Solutions of Periodic System

Optimal upper and lower bounds for multiplitic of coexistence states and conditions for existence of coexistence states we consider the system

$$u_t = u_{xx} + u[a(t,x) - b(t,x)u + c(t,x)v], \quad (3.1)$$
$$v_t = v_{xx} + v[d(t,x) + e(t,x)u - f(t,x)v], \quad (3.2)$$

where it is only assume that the functions a, b, c, d, e and f are positive continuous, and T-periodic on $\Omega \times \mathbb{R}$ under certain conditions on $a, \ldots, f$ we shall obtain upper and lower bounds for the components of coexistence states with more regularity assumptions on $a, \ldots, f$ we shall show in a following that these also imply the existence of coexistence states.

Lemma 2 *Assume that* g *and* k *are continuous and* $T-periodic$ *on* $\Omega \times \mathbb{R}, \partial\Omega$ *is of class* $C^2, w \in C^{2.1}(\Omega \times \mathbb{R}) \cap C^{1.0}(\overline{\Omega} \times \mathbb{R}), k(t,x) > 0, w(t; x) > 0$ *on* $\Omega \times \mathbb{R}$, *w is a solution of*:

$$\frac{\partial w}{\partial t} = k\Delta w + gw \quad (3.3)$$

on $\Omega \times \mathbb{R}, w(x, t + T) \equiv w(t,x)$ *and* w *satisfies the boundary condition*

$$\left.\frac{\partial w}{\partial v}\right|_{\partial\Omega\times[0,T]=0} = 0 \quad (3.4)$$

Then there exist points (t_1, x_1) *and* (t_2, x_2) *in* $\Omega \times \mathbb{R}$ *such that*

$$w(t_1, x_1) = w_l, g(t_1, x_1) \leq 0 \quad (3.5)$$
$$w(t_2, x_2) = w_m, g(t_2, x_2) \geq 0 \quad (3.6)$$

$$where\ w_l = min_{\Omega\times[0,T]}\, w(t,x);\ w_m = max_{\Omega\times[0,T]}\, w(t,x)$$

Proof: Suppose that there exists $(t_1,x_1)\in\Omega\times\mathbb{R}$ such that $w(t_1,x_1)=w_l$ then $w_t(t_1,x_1)=0$ and $\Delta w(t_1,x_1)\geq 0$. Therefore from (3.3) we see that $g(t_1,x_1)w_l\leq 0$ and hence $g(t_1,x_1)\leq 0$. If w does not assume the value w_l any where on open set $\Omega\times\mathbb{R}$ then there exists $(t_1,x_1)\ in\ \partial\Omega\times\mathbb{R}$ such that $w(t_1,x_1)=w_l$ if it were the case that $g(t_1,x_1)>0$then there would exist an open ball D centered at t_1,x_1 such that $g(t_1,x_1)>0$ on $D\cap(\Omega\times\mathbb{R})$ since $w(t,x)>w(t_1,x_1)$ for $(t,x)\in D\cap(\Omega\times\mathbb{R}),\partial\Omega$ is smooth and $k\Delta w - w_t = -gw<0$ at points in $D\cap(\Omega\times\mathbb{R})$ that $\frac{\partial w}{\partial v}(t_1,x_1)<0$which contradicts (3.4) this contradiction shows that $g(t_1,x_1)\leq 0$ if there exists a point (t_2,x_2) in $\Omega\times\mathbb{R}$ such that $w(t_2,x_2)=w_m$ then it follows from (3.3) and the same reasoning as used above that $g(t_2,x_2)\geq 0$ If there exists (t_2,x_2) in $\partial\Omega\times\Omega$ such that $w(t,x)<w(t_2,x_2)$ for all (t,x) in $\Omega\times\mathbb{R}$ and if it were the case that $g(t_2,x_2)<0$ we would have $k\Delta w - w_t>0$ for points in $\Omega\times\mathbb{R}$ near t_2,x_2 in this case the maximum principle would imply that $\frac{\partial w}{\partial v}(t_2,x_2)>0$ contradicting (3.4) show that $g(t_2,x_2)\geq 0$

Theorem 3.1

$Assume\ that\ b_l>\frac{e_m c_m}{f_l}$ (3.7)

$And\ \partial\Omega\ is\ of\ class\ C^2. If\ u,v\in C^{2.1}(\Omega\times\mathbb{R})\cap C^{1.0}(\bar{\Omega}\times\mathbb{R}),(u,v)\ is\ a\ solution\ of$ (3.1) − (3.2) on $(\Omega\times\mathbb{R}), u(t,x)$ and $v(t,x)\ are\ periodic\ solution\ and\ are\ T-periodic\ in\ t, u(t,x)>0\ and\ v(t,x)>0\ for\ (x,t)\in(\Omega\times\mathbb{R}), u$ and $v\ satisfy\ the\ boundary\ conditions$:

$$\left.\frac{\partial u}{\partial v}\right|_{\partial\bar{\Omega}\times[0,T]}=\left.\frac{\partial v}{\partial v}\right|_{\partial\bar{\Omega}\times[0,T]}=0 \qquad (3.8)$$

$Then\ for\ (x,t)\in(\Omega\times\mathbb{R})$:

$$\frac{a_l f_m + d_l c_l}{b_m f_m - c_l e_l}\leq u(t,x)\leq\frac{f_l a_m + c_m d_m}{b_l f_l - e_m c_m}, \qquad (3.9)$$

$$\frac{a_l e_l + b_m d_l}{b_m f_m - c_l e_l}\leq v(t,x)\leq\frac{e_m a_m + b_l d_m}{b_l f_l - e_m c_m}, \qquad (3.10)$$

Proof: Suppose that u and v are as in the statement of the theorem from $lemma$ (2) and the equation (3.1) we suppose the existence of $(t_1,x_1)\ in\ (\Omega\times\mathbb{R})$ such that

$$u(t_1,x_1)=u_l=\min_{t\in\mathbb{R},x\in\bar{\Omega}}u(t_1,x_1)$$

And

$$a(t_1,x_1) - b(t_1,x_1)u(t_1,x_1)+c(t_1,x_1)v(t_1,x_1)\leq 0$$

$$a_l - b_m u_l + c_l v_l\leq 0\ (3.11)$$

similarly from $lemma$ (2) and equation (3.2) we suppose the existence of $(t_2,x_2)\ in\ (\Omega\times\mathbb{R})$ that

$$v(t_2,x_2)=v_l=\min_{t\in\mathbb{R},x\in\bar{\Omega}}u(t_2,x_2)$$

And

$$d(t_2,x_2)+e(t_2,x_2)u(t_2,x_2)-f(t_2,x_2)v(t_2,x_2)\leq 0$$

$$d_l + e_l u_l - f_m v_l\leq 0\ (3.12)$$

we multiply (3.11) by e_l and (3.12) by b_m then we obtain

$$a_l e_l - b_m e_l u_l + c_l e_l v_l\leq 0$$
$$d_l b_m + b_m e_l u_l - b_m f_m v_l\leq 0$$

Then

$$a_l e_l + d_l b_m + (c_l e_l - b_m f_m)v_l\leq 0$$
$$(b_m f_m - c_l e_l)v_l\geq a_l e_l + d_l b_m$$
$$v_l\geq\frac{e_l a_l + b_m d_l}{b_m f_m - e_l c_l}\ (3.13)$$

and multiply (3.11) by f_m and (3.12) by c_l then we obtain

$$a_l f_m - b_m f_m u_l + c_l f_m v_l\leq 0$$
$$d_l c_l + c_l e_l u_l - c_l f_m v_l\leq 0$$

Then

$$a_l f_m + d_l c_l + (c_l e_l - b_m f_m)u_l\leq 0$$
$$(b_m f_m - c_l e_l)u_l\geq a_l f_m + d_l c_l$$
$$u_l\geq\frac{f_m a_l + d_l c_l}{b_m f_m - e_l c_l}\ (3.14)$$

from $lemma$ (2) and the equation (3.1)we suppose the existence of $(t_3,x_3)\ in\ (\Omega\times\mathbb{R})$such that

$$u(t_3,x_3)=u_m=\max_{t\in\mathbb{R},x\in\bar{\Omega}}u(t,x)$$

and $a(t_3,x_3)-b(t_3,x_3)u_m + c(t_3,x_3)v(t_3,x_3)\geq 0$

$$a_m - b_l u_m + c_m v_m\geq 0\ (3.15)$$

similarly from $lemma$ (2) and equation (3.2) we suppose the existence of $(t_4,x_4)\ in\ (\Omega\times\mathbb{R})$ such that

$$v(t_4,x_4)=v_m=\max_{t\in\mathbb{R},x\in\bar{\Omega}}v(t,x)$$

and $d(t_4,x_4)+e(t_4,x_4)u(t_4,x_4)-f(t_4,x_4)v(t_4,x_4)\geq 0$

$$d_m e_m u_m - f_l v_m\geq 0\ (3.16)$$

we multiply (3.15) by e_m and (3.16) by b_l then we obtain

$$e_m a_m - e_m b_l u_m + e_m c_m v_m\geq 0$$
$$b_l d_m + b_l e_m u_m - b_l f_l v_m\geq 0$$

then

$$e_m a_m + d_m b_l + (e_m c_m - b_l f_l)v_m\geq 0$$

$$e_m a_m + d_m b_l\geq(b_l f_l - e_m c_m)v_m$$

$$\frac{e_m a_m + d_m b_l}{b_l f_l - e_m c_m}\geq v_m\ (3.17)$$

we multiply (3.15) by $f_l$and (3.16) by c_m then we obtain

$$f_l a_m - b_l f_l u_m + f_l c_m v_m\geq 0$$

$$c_m d_m + c_m e_m u_m - f_l c_m v_m\geq 0$$

and then

$$f_l a_m + c_m d_m + (c_m e_m - b_l f_l)u_m\geq 0$$

$$f_l a_m + c_m d_m\geq(b_l f_l - c_m e_m)u_m$$

$$\frac{f_l a_m + c_m d_m}{b_l f_l - c_m e_m}\geq u_m\ (3.18)$$

Since $u_l\leq u(t,x)\leq u_m$ and $v_l\leq v(t,x)\leq v_m$ from inequalities (3.13), (3.14), (3.17) and (3.18) we obtain:

$$\frac{f_m a_l + c_l d_l}{b_m f_m - c_l e_l}\leq u(t,x)\leq\frac{f_l a_m + c_m d_m}{b_l f_l - c_m e_m}$$

$$\frac{e_l a_l + b_m d_l}{b_m f_m - c_l e_l} \le v(t,x) \le \frac{e_m a_m + b_l d_m}{b_l f_l - c_m e_m}$$

this proves theorem (3.1)

4. Asymptotic Behavior of Periodic System

Theorem 4.1

If $b_l > \frac{c_m e_m}{f_l}$ (4.1)
Then there exist pairs $(\hat{u},\hat{v})$ and $(\hat{u}^*,\hat{v}^*)$ with components in $C^{2+\infty,1+\infty}(\Omega\times\mathbb{R})$ such that the components of both pairs are strictly positive and T-periodic in t each pair is a solution of (1.1)and satisfies the boundary conditions (1.2).
Moreover

$$\hat{u}(t,x) \le \hat{u}^*(t,x); \hat{v}(t,x) \le \hat{v}^*(t,x)\ on\ \Omega\times\mathbb{R}$$

and if (u,v) is a solution of the initial boundary value problem given by $(1.1) - (1.2)$ with $u(0,x) = \Phi(x)$; $v(0,x) = \Psi(x)$ such that $\Phi,\Psi \in C^{1+\infty}(\Omega)$

$$\Phi(x) \ge 0;\ \Psi(x) \ge 0;\ \Phi(x) \not\equiv 0;\ \Psi(x) \not\equiv 0$$

and:

$$\left.\frac{\partial\Phi}{\partial v}\right|_{\partial\Omega} = \left.\frac{\partial\Psi}{\partial v}\right|_{\partial\Omega} = 0$$

then for any $\varepsilon > 0$:

$$\hat{u}(t,x) - \varepsilon < u(t,x) < \hat{u}^* + \varepsilon$$
$$\hat{v}(t,x) - \varepsilon < v(t,x) < \hat{v}^* + \varepsilon$$

for x in Ω and all sufficiently large t. If a,b,c,d,e and f are function of t alone.

Then $\hat{u} \equiv \hat{u}^*$; $\hat{v} \equiv \hat{v}^*$ and $\hat{u},\hat{v}$ are functions of t alone.

Proof: Choose constant k_1,k_2 such that

$$a_m - b_l k_1 + c_m k_2 < 0, (4.2)$$

$$d_m + e_m k_1 - f_l k_2 < 0, (4.3)$$

using (4.1), k_1 and k_2 exist, then choose $0 < \delta_1 < k_1, 0 < \delta_2 < k_2$ such that

$$a_l - b_m\delta_1 + c_l\delta_2 > 0, (4.4)$$
$$d_l + e_l\delta_1 - f_m\delta_2 > 0, (4.4)$$

in fact if $\delta_1 < \frac{a_l}{b_m}$; $\delta_2 < \frac{d_l}{f_m}$; δ_1,δ_2 are suitable. It is obvious that (k_1,k_2) and (δ_1,δ_2) are periodic upper and lower solutions. There exist two pairs periodic solutions of original periodic boundary value problem $(1.2) - (2.1)$ $(\hat{u},\hat{v}),(\hat{u}^*,\hat{v}^*)$ and:

$$\delta_1 \le \hat{u} \le \hat{u}^* \le k_1, (4.6)$$
$$\delta_2 \le \hat{v} \le \hat{v}^* \le k_2, (4.7)$$

we consider following initial boundary value problem $(1.2) - (2.1)$ with:

$$u(x,t_0) = \delta_1;\ v(x,t_0) = \delta_2, (4.8)$$
$$u(x,t_0) = k_1;\ v(x,t_0) = k_2, (4.9)$$

Denote: (u_1,v_1) and (u_2,v_2) are the corresponding solutions of $(1.2) - (2.1)$, (4.8) and $(1.2) - (2.1)$, (4.9) then $\lim_{t\to\infty}(u_1,v_1) = (\hat{u},\hat{v})$ and $\lim_{t\to\infty}(u_2,v_2) = (\hat{u}^*,\hat{v}^*)$.
Finally we consider initial boundary value problem for $u_0(x) \ge 0$; $v_0(x) \ge 0$ and $u_0(x) \not\equiv 0$; $v_0(x) \not\equiv 0$ by strong maximum principle

$$u_0(t_0 + 1,x) > 0, v_0(t_0 + 1,x) > 0\ (4.10)$$

we choose (δ_1,δ_2) and (k_1,k_2) such that

$$\delta_1 \le u_0(t_0 + 1,x) \le k_1;\ \delta_2 \le v_0(t_0 + 1,x) \le k_2\ (4.11)$$

we obtain following comparison relation:

$$u_1(t-1,x) \le u(t,x) \le u_2(t-1,x), v_1(t-1,x) \le v(t,x) \le v_2(t-1,x), t \ge t_0 + 1\ (4.12)$$

Therefore we have

$$\hat{u}(t,x) \le \liminf_{t\to\infty} u(t,x), \limsup_{t\to\infty} u(t,x) \le \hat{u}^*(t,x), (4.13)$$

and

$$\hat{v}(t,x) \le \liminf_{t\to\infty} v(t,x), \limsup_{t\to\infty} v(t,x) \le \hat{v}^*(t,x). (4.14)$$

For uniqueness:

$$\int_0^T [a(t) - b(t)\hat{u}(t) + c(t)\hat{v}(t)]dt = \int_0^T [a(t) - b(t)\hat{u}^*(t) + c(t)\hat{v}^*(t)]dt.$$

By means of $\int_0^T \frac{\hat{u}_t}{\hat{u}}dt = \int_0^T \frac{\hat{u}^*_t}{\hat{u}^*}dt$

$$\int_0^T b(t)(\hat{u}^*(t) - \hat{u}(t))dt = \int_0^T c(t)(\hat{v}^*(t) - \hat{v}(t))dt = 0\ (4.15)$$

If: $(\hat{u}^*(t) \ge \hat{u}(t)$ and $(\hat{u}^*(t) \not\equiv \hat{u}(t)$ we have

$$\frac{b_l}{c_m} \le \frac{\int_0^T(\hat{v}^*(t) - \hat{v}(t))dt}{\int_0^T(\hat{u}^*(t) - \hat{u}(t))dt}\ (4.16)$$

similarly we have:

$$\frac{e_m}{f_l} \le \frac{\int_0^T(\hat{v}^*(t) - \hat{v}(t))dt}{\int_0^T(\hat{u}^*(t) - \hat{u}(t))dt}\ (4.17)$$

it follows from $(4.16) - (4.17)$

$$\frac{b_l}{c_m} \le \frac{e_m}{f_l}\ (4.18)$$

it is contradiction with (4.1), therefore the uniqueness holds.

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