On Non-Homogeneous Bi-Quadratic Diophantine Equation $4(x^2+y^2)-7xy = 19z^4$

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Abstract: Five different methods of the non-zero integral solutions of the homogeneous biquadratic Diophantine equation with five unknowns $4(x^2+y^2)-7xy = 19z^4$ are determined. Some interesting relations among the special numbers and the solutions are exposed.

Keywords: Quadratic, non-homogenous, integer solutions, special numbers, polygonal, and pyramidal numbers

Mathematics Subject Classification: 11D09

Notations used:
- $T_{m,n}^r$: Polygonal number of rank n with sides m.
- $p_n^m$: Pyramidal number of rank m with side n.
- $G_n$: Gnomonic number of rank n.
- $f_{4,3}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Triangle.
- $f_{4,4}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Square.
- $f_{4,5}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Pentagon.
- $f_{4,6}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Hexagon.
- $f_{4,7}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Heptagon.
- $f_{4,8}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Octagon.

1. Introduction

The number theory is the queen of Mathematics. In particular, the Diophantine equations have a blend of attracting interesting problems. For an extensive review of variety of problems, one may refer to [3-12]. In 2014, Jayakumar. P, Sangeetha. K, [12] have published a paper in finding the integer solutions of the homogeneous Biquaratic Diophantine equation $(x^3 - y^3)z = (W^2 - P^2)R^4$. In 2015, Jayakumar. P, Meena.J [14, 15] published two papers in finding integer solutions of the homogeneous Biquaratic Diophantine equation $(x^4 - y^4) = 26 (z^2 - w^2) R^2$ and $(x^4 - y^4) = 40 (z^4 - w^4) R^4$. Inspired by these, in this work, we are observed another interesting five different methods of the non-zero integral solutions of the non homogeneous biquadratic Diophantine equation with three unknowns $4(x^2+y^2)-7xy = 19z^4$. Further, some elegant properties among the special numbers and the solutions are exposed.

2. Description of Method

Consider biquadratic Diophantine equation $4(x^2+y^2)-7xy = 19z^4$ (1)

Introduce the linear transformations
- $x = u + v, y = u - v$ (2)

Using (2) in (1), this gives to
- $u^2 + 15v^2 = 19z^4$ (3)

We solved (3) through various choices and the different methods of solutions of (1) are obtained as follows.

2.1 Method: I

Consider (3) as $u^2 + 15v^2 = 19z^4$ and take it as in the form of ratio as
- $\frac{u + 2z^2}{15(z^2 + v)} = \frac{z^2 - v}{u - 2z^2} = \frac{a}{b}$, $b \neq 0$ (4)

(4) is equivalent to the system of equations as
- $6u - 15av + (2b - 15a)v^2 = 0$ (5)
- $- au - bv + (2a + b)z^2 = 0$ (6)

By the cross multiplication method, the above equations yields as
- $u = 30a^2 - 2b^2 + 30ab$
- $v = 15a^2 + b^2 + 4ab$(7)
- $z^2 = 15a^2 + b^2$

If we take a = 2pq, b = 15p² - q² in (7) and using (2), then we find
- $x = x (p, q) = - 225p^4 - q^4 + 90p^2q^2 + 1020p^3q - 68pq^3$
Instead of (4) take the form of ratio as
\[
\frac{u + 2z^2}{z^2 - v} = \frac{15(z^2 + v)}{a^2 - b^2}, \quad b \neq 0
\]
(8)
The procedure following is same as the method -I, the relating integer solutions to (1) are found as
\[
x = x(p, q) = 675p^3 + 3q^3 - 270p^2q^2 + 780pq^3 - 52pq^4
\]
y = y(p, q) = 225p^3 + q^3 + 90p^2q^3 + 1020pq^3 - 68pq^4
z = z(p, q) = 15p^3 + q^2
(9)

Observations:-
1. \(x(p, q) = 36 f_{4,9}^p + 128 P^5_p + 221T_a + G_{15a} = 0 \) (Mod 2)
2. y(p, q) - 16200 \( f_{4,9}^q \) + 3375T_{qa} + 20040 \( P^5_q - 5700T_{aq} - G_{132a} = 0 \) (Mod 2)
3. y(p, q) - 24 \( f_{4,9}^p + 4T_{aq} + 164 P^5_q + 15T_{aq} - G_{311a} = 0 \) (Mod 2)

2.3 Method: III
Let us take
\[
19 = (2 + i\sqrt{3}) (2 - i\sqrt{3})
\]
(10)
Take z as
\[
z = z(a, b) = a^2 + 15b^2
\]
Using (9) and (10) is (3) and applying factorization process, define
\[
u + i\sqrt{3}v = (2 + i\sqrt{3}) (a + i\sqrt{3}b)^3
\]
This gives us
\[
u = 2a^3 + 450b^4 - 180a^2b^2 - 60ab^3 + 900ab^4
\]
\[
v = a^2 + 225b^4 - 90a^2b^2 + 8a^3b - 120ab^3
\]
(11)
Using (11) in (2), the relating integer solutions to (1) are found as
\[
x = x(a, b) = 5a^3 + 675b^4 - 270a^2b^3 - 52ab^2 + 780ab^3
\]
y = y(a, b) = a^2 + 225b^4 - 90a^2b^2 - 68a^3b + 1020ab^3
z = z(a, b) = a^2 + 15b^2
(12)
Observations: 1. x(1, A) + y(1, A) - 10800 \( f_{4,9}^A + 3600 P^5_A \) + 3060A + G_{506a} = 1 (Mod 2)
2. x(1, A) - y(1, A) - 10800 \( f_{4,9}^A + 1800T_{A,2} + 13080 P^5_A \) - 3210T_{A,4} - G_{372a} = 1 (Mod 2)
3. x(1, A) - 16200 \( f_{4,9}^A + 6540 P^5_A + 4425T_{A,4} + G_{2051a} = 0 \) (Mod 2)
4. y(1, A) - 1350 \( f_{4,9}^A - 690 P^5_A + 885T_{A,4} + G_{34a} = 0 \)
5. 6z(1, 0) is a Nasty Number.

2.4 Method: IV
In place of (9) take 19 as
\[
19 = \left(\frac{(17 - i\sqrt{3})(17 + i\sqrt{3})}{2}\right)
\]
(13)
The procedure following is same as the method -III, the relating integer solutions to (1) are found as
\[
u = \frac{1}{4} \left[17a^4 + 3825b^5 + 1530a^2b - 900ab^3 - 60a^2b^2\right]
\]
(14)
\[
v = \frac{1}{4} \left[a^2 + 225b^4 - 90a^2b^2 - 1020ab^3 + 68a^3b\right]
\]
In true of (2), the values x and y are
\[
x = \frac{1}{4} \left[18a^4 + 4050b^5 - 1620a^2b^2 - 120ab^3 + 8a^3b\right]
\]
(15)
\[
y = \frac{1}{4} \left[16a^4 + 3600b^4 - 1440a^2b^3 + 19200ab^3 - 128a^3b\right]
\]
(16)
Since our intention is to find integer solutions, taking a as 4a and b as 4b in (4),(15) and (16), the relating parametric integer values of (1) are found as
\[
x = x(A, B) = 576A^4 + 129600B^4 - 51840A^2B^2 + 256A^3B^3 - 3840AB^3
\]
y = y(A, B) = 512A^4 + 115200B^4 - 46080A^2B^2 - 4096A^3B + 61440AB^3
z = 16A^2 + 240B^2
Observations: 1. x(1, n) + y(1, n) - 1468800 \( f_{4,9}^n + 1353600 P^5_n - 89280T_{A,n} + G_{512a} = 1 \) (Mod 2)
2. x(1, n) - y(1, n) - 345600 \( f_{4,9}^n - 418560 P^5_n - 28800T_{A,n} + G_{122a} = 1 \) (Mod 2)
3. y(1, q) - 13824 \( f_{4,9}^q + 2304T_{aq} + 15616 P^5_q + 48064T_{A,q} + G_{1344a} = 1 \) (Mod 2)
4. y(1, q) - 6144 \( f_{4,9}^q + 12288 P^5_q + 42496T_{A,q} - G_{5020a} = 0 \) (Mod 5)
5. x(1, A) is a perfect square.

2.5 Method: V
Let us take (3) as
\[
u^2 + 15v^2 = 19z^4 \star 1
\]
Take 1 as \(1 = \left(1 + i\sqrt{3}\right) \left(1 - i\sqrt{3}\right)\)
\]
(17)
Using (9) and (10) in (13) and applying factorization process, define
\[
u + i\sqrt{3}v = (2 + i\sqrt{3}) (a + i\sqrt{3}b)^3 \left(1 + i\sqrt{3}\right)
\]
It furnishes us
\[
u = \frac{1}{4} \left[-13a^4 - 2925b^5 + 1170a^2b^2 + 2700ab^3 - 180a^2b^2\right]
\]
(18)
\[
v = \frac{1}{4} \left[3a^4 + 675b^4 - 270a^2b^3 + 780ab^3 - 52ab^2\right]
\]
In sight of (2), the values of x and y as


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