Chebyshev Best Approximation of (p,q)- Type and Lower (p,q)- Type of Entire Functions of Several Complex Variables

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Abstract: ***This paper deals with the characterization of generalized*** $(p,q)-$ ***type and generalized lower*** $(p,q)-$ ***type with respect to proximate order of an entire function*** $f:C^2 \to C$ ***in terms of the Chebyshev best approximation to*** f ***on compact set*** $E \subset C^2$ ***by polynomials. In this paper we want to establish formulae for lower*** (p,q)***- type of entire functions of two complex variables with index pair -*** (p,q)***.***

Keywords: Chebyshev approximation, polynomials , generalized, bounded, proximate order

1. Introduction

Let E be a bounded closed set in the space C^2 of two complex variables $z=(z_1,z_2)$ with the norm

$$\|f\|_E = \sup\{|f(z)| : z \in E\}$$

For a function f defined and bounded on E.

Let P_ν denote the set of all polynomials in z of degree $< V$. Set

$$E_\nu(f,E) = \inf\{\|f-p\|_E : p \in P_\nu\}.$$

The following theorem for single complex variable was proved by Winiarski [1].

Theorem 1: A function f defined and bounded on a closed set E with a positive transfinite diameter d, can be continued to an entire function f of order $\rho(0<\rho<\infty)$ and of type $\sigma(0<\sigma<\infty)$ if and only if

$$(1.1) \quad \limsup_{\nu\to\infty} \nu^{1/\rho}(E_\nu(f,E))^{1/\nu} = d(e\sigma\rho)^{1/\rho}.$$

It has been noted that in two or more than two complex variables, the type and lower type of $f(z)$ can not be characterized by means of the measure of the Chebyshev best approximation to f on E by polynomials of degree $\le \nu$ with respect to all variable. Due to this fact Kumar D. [2] considered the measures $E_k^*(f,E)$, $k=(k_1,k_2)$ of the Chebyshev best approximation to f in $E=E^{(1)}\times E^{(2)}$ by polynomials of degree $\le k_j$ with respect to the j^{th} variable, $j=1,2,$ where E_j is bounded closed set with a positive transfinite diameter $d_j = d(E^j)$ in the complex z_j plane. He extended the above theorem for two complex variables and to estimate the slow and fast growth of entire functions. This theorem also have been extended to $(p,q)-$ scale introduced by Juneja et al. ([3],[4]). But these results are inadequate for comparing the growth of those entire functions which are of same $(p,q)-$ order but of infinite $(p,q)-$ types. To refine this scale we shall obtain $(p,q)-$ type and lower $(p,q)-$ type with respect to proximate order of index pair (p,q), for integers p and q such that $1\le q\le p$. Analogous results for generalized lower $(p,q)-$ type also have been studied.

Let D be a complex Banach space with norm $\|\cdot\|$. Let $f:C^2\to D$ be an entire function and $P_k = P_k(C^2,D), k=(k_1,k_2)$ be the set of all polynomials $p:C^2\to D$ of degree $\le k_j$, with respect to j^{th} variable, respectively, $j=1,2.$

Let E be a compact set in C^2 and let $f:E\to D$ be a function defined and bounded on E. Set

$$E_k^*(f,E) = \inf\{\|f-p\|_E : p\in P_k\}.$$

Let $E=E^{(1)}\times E^{(2)}$, when $E^{(j)}(j=1,2)$ is a compact set in C containing infinitely many different points. Let $n_j^{k_j} = (n_{j0},\ldots,n_j k_j), j=1,2$, be a system of k_j+1 extremal points of E_j (see [2]).The polynomial

$$L_k(z) = \sum_{u_1,u_2=0}^{k_1,k_2} (n_{1u_1},\eta_{2u_2}) L^{(u_1)}(z_1) L^{(u_2)}(z_2)$$

is the Lagrange interpolation polynomial for f with nodes $n_1^{(k_1)} \times n_2^{(k_2)}$ of degree $\le k_j$ with respect to the j^{th} variable. Here

$$L^{(u_j)}(z_j) = L^{(u_j)}(z_j - E_j) = \frac{(z_j - n_{j^0})\cdots}{(n_j u_j - n_{j^0})\cdots}\bigg|\frac{(z_j n_j k_j)}{(n_j u_j - n_j k_j)}$$

where $|u_j$ means that the factor u_j is omitted. The inequality
(1.2)

$$E_k^*(f,E) \le \|f - L_k\|_E \left(1 + \prod_{j=1}^{2}(k_j+1)\right) E_k^*(f,E)$$

can be proved in a similar manner as Lemma 1.1 of [2].

Definition 1: An entire function defined on C^2 is said to be $(p,q)-$order $\rho(p,q)$ and if $(b < \rho(p,q) < \infty$ $(p,q)-$type $\tau(p,q)$ and lower $(p,q)-$type $t(p,q)$ if

$$\rho(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} \rho(r,f)}{\log^{[q]} r}$$

$$\begin{matrix}\tau(p,q)\\ t(p,q)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]} S(r,f)}{(\log^{[q-1]} r)^{\rho(p,q)}}, O < t(p,q) \le \tau(p,q) \le \infty$$

where

$$S(r,f) = \sup_{|z|=r} \{\|f(z)\|\} \;\forall\; r \in R^+,\ \log^{|m|} x = \exp^{|-m|} x < \infty$$

with $\log^{|0|} x = \exp^{|0|} = x, b = 1$ if $p = q$ and $b = 0$ if $p > q$.

Definition 2: The generalized $(p,q)-$type τ^* and generalized lower $(p,q)-$type t^* of $f(z)$ are defined as

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]} S(r,f)}{(\log^{[q-1]} r)^{\rho(r)}} = \begin{matrix}\tau^*(p,q) \equiv \tau^*\\ t^*(p,q) \equiv t^*\end{matrix},$$

The function $\rho(r)$ is said to be proximate order and lower proximate order of the given function f if τ^* and t^* are nonzero and finite respectively.

Now we shall prove the main results.

Theorem 2: Let $f(z_1,z_2) = \sum_{m_1,m_2=0}^{\infty} a_{m_1,m_2} z_1^{m_1} z_2^{m_2}$ be an entire function of index-pair (p,q). If $0 < U < \infty$, the function $f(z_1,z_2)$ is of $(p,q)-$order $\rho(p,q) = (\rho_1(p,q), \rho_2(p,q)) > (b,b)$ and generalized $(p,q)-$type $\tau^*(p,q)$ $= (\tau_1^*(p,q), \tau_2^*(p,q))$ $(0,0)$, with respect to proximate order $\rho(r)$. Then

$$\tau^*(p,q) = M'U$$

where

$$U \equiv U(p,q) = \limsup_{\min(k_j)\to\infty} \left[\frac{\varphi(\log^{[p-2]}(k))}{\log^{[q-1]} E_k^*(f,E)^{-1/k}}\right]^{\rho(p,q)-A}.$$

$$M' = \begin{cases} \dfrac{(\rho(2,2)-1)^{\rho(2,2)-1}}{\rho(2,2)^{\rho(2,2)}} & \text{if } (p,q) = (2,2) \\ 1/e\rho(2,1)d^{\rho(2,1)} & \text{if } (p,q) = (2,1) \\ 1 & \text{otherwise.} \end{cases}$$

$$A = \begin{cases} 1 & \text{for}\,(p,q) = (2,2) \\ 0 & \text{otherwise}; \end{cases}$$

$$b = \begin{cases} 1 & \text{for}\,(p,q) = (2,2) \\ 0 & \text{otherwise}; \end{cases}$$

$k = (k_1,k_2)$ and $d_j = d(E^j) > 0\ (j = 1,2)$ are the transfinite diameters of E^j.

Proof. From [2], eq. 1.10], we have

$$\|f - L_k\|_E \le \lambda \frac{M(r,f)}{r^k}(d\, e^{\epsilon})^k, \quad (2.1)$$

for $r > r^{(1)} = (r_1^{(1)}, r_2^{(1)}), k > k^{(1)} = (k_1^{(1)}, k_2^{(1)})$,

where $\lambda = \lambda_1, \lambda_2, \epsilon = (\epsilon_1, \epsilon_2), e^{\epsilon} = (e^{\epsilon_1}, e^{\epsilon_2})$.

Let $k(p,q) = (k_1(p,q), k_2(p,q)) > \tau^*(p,q)$. By definite of generalized $(p,q)-$type of $f(z_1,z_2)$ there exists an $r^{(2)} > r^{(1)}$ such that

$$\frac{\log^{[p-1]} S(r,f)}{(\log^{[q-1]} r)^{\rho(r)}} \le K(p,q) \text{ for } r > r^{(2)},$$

or

$$S(r,f) \le \exp^{[p-1]}[K(p,q)(\log^{[q-1]} r)^{\rho(r)}]. \quad (2.2)$$

For $(p,q) = (2,1)$, using (2.2) with (2.1) we get

$$\|f - L_k\|_E \le \lambda(d\, e^{\epsilon})^k \left[\frac{e^{k(2,1)} r^{\rho(r)}}{r^k}\right]. \quad (2.3)$$

Let $k^{(2)} > k^{(1)}$ such that $\dfrac{k_j}{r_j \rho_j(2,1)} > r_j^{(\rho(r))}$ for $j = 1,2,\ k > k^{(2)}$.

Choosing

$$r = \left[\varphi\left(\frac{k_1}{K_2(2,1)\rho_2(2,1)}\right), \varphi\left(\frac{k_2}{K_2(2,1)\rho_2(2,1)}\right)\right]$$

in (2.3) we get

$$\|f - L_k\|_E < \lambda(d\, e^{\epsilon})^k \frac{(e^{\phi} k(2,1)\,\rho(2,1))^{k/\rho(2,1)}}{[\varphi(k)]^k}$$

$$\leq \lambda d^k \left(\frac{(e\tau^*(2,1)\,\rho(2,1))^{k/\rho(2,1)}}{[\varphi(k)]^k} \right) (e^{\epsilon-\delta/k})^k \text{ for } k > k^{(2)},\ \delta = (\delta_1, \delta_2)$$

which gives

$$\left[\varphi(k)\left(\|f - L_k\|_E\right)^{1/k}\right]^{\rho(2,1)} \leq e\rho(2,1)\,\tau^*(2,1)\, d^{\rho(2,1)} \lambda^{\rho(2,1)/k} (e^{\epsilon-\delta/k})^{\rho(2,1)}$$

or

$$\left[\frac{\varphi(k)}{\log\|f - L_k\|_E^{-1/k}}\right]^{\rho(2,1)} \leq \frac{\tau^*(2,1)}{M'} \text{ as } k \to \infty. \quad (2.4)$$

For $(p,q) = (2,2),$ in view of (2.2), we have

$$\|f - L_k\|_E \leq \lambda (d\, e^{\epsilon})^k \exp[k(2,2)(\log r)^{\rho(r)}]\left(\frac{1}{r^k}\right). (2.5)$$

Let $k^{(2)} > k^{(1)}$ such that

$$r^{\rho(r)} = \frac{k_j}{k(2,2)\,\rho_j(2,2)} > r_j^{\rho(r)}, (j = 1,2),\ k > k^{(2)}.$$

Choosing

$$r = \left[\exp\varphi\left(\frac{k_1}{K_1(2,2)\rho_2(2,2)}\right), \exp\varphi\left(\frac{k_2}{K_2(2,2)\rho_2(2,2)}\right)\right]$$

in (2.5), we get

$$\|f - L_k\|_E \leq \frac{\lambda(de^{\epsilon})\left\{\exp\left[\frac{k}{\rho(2,2)} \cdot \frac{\varphi(k)}{(K(2,2)\rho(2,2))^{1/(\rho(2,2)-1)}}\right]\right\}}{\left[\exp\left[\frac{\varphi(k)}{(K(2,2)\rho(2,2))^{1/(\rho(2,2)-1)}}\right]\right]^k}$$

or

$$\log\|f - L_k\|_E \leq \log\lambda + \frac{k\varphi(k)}{K(2,2)\rho(2,2))^{\rho/\rho-1}} - \frac{k\varphi(k)}{[K(2,2)\rho(2,2)]^{1/(\rho(2,2)-1)}}.$$

$$\left[\log\|f - L_k\|_E^{-1/k}\right]^{\rho(2,2)-1} \geq \frac{[\varphi(k)]^{\rho(2,2)-1}}{K(2,2)}\left(\frac{(\rho(2,2)-1)^{\rho(2,2)-1}}{\rho(2,2)^{\rho(2,2)}}\right)[1 - O(1)]^{\rho(2,2)-1}$$

or

$$\frac{K(2,2)}{M'} \geq \limsup_{\min(k_j)\to\infty}\left[\frac{\varphi(k)}{\log\|f - L_k\|_E^{-1/k}}\right]^{\rho(2,2)-1}. (2.6)$$

Now we consider the case when $(p,q) \neq (2,1)$ and $(2,2)$ i.e. $3 \leq q \leq p < \infty,$ let $k^{(2)} > k^{(1)}$ such that

$$\exp^{[q-1]}\left[\frac{\log^{[p-2]}(k_j / K(p,q)\rho_j(p,q))}{K(p,q)}\right] > r_j^{\rho(r)}$$

for $k > k^{(2)},\ j = 1,2.$

Choosing

$$r = \left[\exp^{[q-1]}\left[\varphi\left(\frac{\log^{[p-2]}(k_1 / K_1(p,q)\rho_1(p,q)}{K_1(p,q)}\right)\right], \exp^{[q-1]}\left[\varphi\left(\frac{\log^{[p-2]}(k_2 / K_2(p,q)\rho_2(p,q)}{K_2(p,q)}\right)\right]\right]$$

in (2.1) and (2.2), we obtain

$$\log\|f - L_k\|_E^{-1/k} \geq \exp^{[q-2]}\left[\varphi\left(\frac{\log^{[p-2]}(k / K(p,q)\rho(p,q)}{K(p,q)}\right)\right][1 - O(1)]$$

for sufficiently large values of k_j's or

$$K(p,q) \geq \left[\frac{\varphi(\log^{[p-2]}(k/K(p,q)\rho(p,q)))}{\log^{[q-1]} \|f - L_k\|_E^{-1/k}} \right]^{\rho(p,q)} + o(1).$$

Proceeding to limits, we get

$$K(p,q) \geq \limsup_{\min(k_j)\to\infty} \left[\frac{\varphi(\log^{[p-2]}(k/K(p,q)\rho(p,q)))}{\log^{[q-1]} \|f - L_k\|_E^{-1/k}} \right]^{\rho(p,q)}. \quad (2.7)$$

Since (2.4), (2.6) and (2.7) are valid for every $K(p,q) = (K_1(p,q), K_2(p,q)) > \tau^*(p,q)$, it follows that

$$\limsup_{\min(k_j)\to\infty} \left[\frac{\varphi(\log^{[p-2]} k)}{\log^{[q-1]} \|f - L_k\|_E^{-1/k}} \right]^{\rho(p,q)-A} \leq \frac{\tau^*(p,q)}{M'}. \quad (2.8)$$

To prove the reverse inequality, let $\tilde{\nu} = (\nu,\nu) \in R^2, \nu = 0;1\cdots$ and in view of Lemma 1.1 of [2] to the function f in the series

$$f(z) = L_{\tilde{0}}(z) + \sum_{\nu=0}^{\infty} (L_{\tilde{\nu}+1}(z) - L_{\tilde{\nu}}(z)) \quad z \in C^2.$$

We get $\|f(z)\| \leq \|L_{\tilde{0}}(z)\| + \sum_{\nu=0}^{\infty} \|L_{\tilde{\nu}+1}(z) - L_{\tilde{\nu}}(z)\|,$

We have the property of extremal function $\varphi(z,E)$ [5]

$$\|p(z)\| \leq \|p\|_E \varphi^{\nu}(z),\ z \in C^2. \quad (2.9)$$

Applying (2.9) in above for every variable separately, we get

$$\|f(z)\| \leq a_0 + 2\sum_{\nu=0}^{\infty} \|f - L_{\tilde{\nu}}\|_E (r/d)^{\tilde{\nu}} \text{ for } z \in E_{r_j}^{(j)}. \quad (2.10)$$

Consider the function

$$g(z) = \sum_{\nu=0}^{\infty} \|f - L_{\tilde{\nu}}\|_E \ z^{\tilde{\nu}}.$$

Since $\lim_{\tilde{\nu}\to\infty} \|f - L_{\tilde{\nu}}\|_E^{1/\tilde{\nu}} = 0$ in view of Lemma 1.1 of [2] it follows that $g(z)$ is entire function. From (2.10) we obtain

$$S(r, f) \leq a_0 + 2g(r/d). \quad (2.11)$$

Now applying Theorem , for each variable separately, we get

$$\frac{\tau^*(p,q)}{M'} \leq \limsup_{\min(\tilde{\nu}_j)\to\infty} \left[\frac{\varphi(\log^{[p-2]} \tilde{\nu})}{\log^{[q-1]} \|f - L_k\|_E^{-1/\tilde{\nu}}} \right]^{\rho(p,q)-A}. \quad (2.12)$$

Using inequality (2.4) with (2.8) and (2.12) together proves the theorem.

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