International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2013): 6.14 | Impact Factor (2015): 6.391

Bayesian Estimation of the Failure Rate Using Extension of Jeffreys" Prior Information with Three Loss Functions

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Abstract: ***The Weibull distribution is widely used in Reliability and life data analysis due to its versatility. We Consider the Constant Shape Bi-Weibull distribution which has been extensively used in the testing and reliability studies of the strength of materials. Studies have been done vigorously in the literature to determine the best method in estimating its Failure Rate. In this paper, we examine the performance of Maximum Likelihood Estimator (MLE) and Bayesian Estimator using Extension of Jeffreys' Prior Information with three Loss functions, namely, the Linear Exponential (LINEX) Loss, General Entropy Loss, and Square Error Loss for estimating the Constant Shape Bi-Weibull Failure time distribution. The results show that Bayesian Estimator using Extension of Jeffreys' Prior under Linear Exponential (LINEX) Loss function in most cases gives the smallest Mean Square Error and Absolute Bias for Failure Rate FR(t) for the given values of Extension of Jeffreys' Prior. An illustrative example is also provided to explain the concepts.***

Keywords: Bayesian method, Constant Shape Bi-Weibull Failure time Distribution, Extension of Jeffreys Prior information, Failure Rate, MLE.

1. Introduction

The Weibull distribution is widely used in Reliability and life data analysis due to its versatility. Depending on the values of the parameters, the Weibull distribution can be used to model a variety of life behaviours. An important aspect of the Weibull distribution is how the values of the shape parameter, $\boldsymbol{\beta}$, and the scale parameter, $\boldsymbol{\sigma}$, affect the characteristics life of the distribution, the shape/slope of the distribution curve, the Reliability Function, and the Failure Rate.

The main purpose of this paper is to compare the traditional Maximum Likelihood Estimation of the Failure Rate of the Constant Shape Bi-Weibull distribution with its Bayesian counterpart using Extension of Jeffreys" Prior Information obtained from Lindley"s approximation procedure with three Loss Functions.

It has been found that this distribution is satisfactory in describing the life expectancy of components that involve fatigue and for assessing the Reliability of bulbs, ball bearings, and machine parts according to [15].The primary advantage of Weibull analysis according to [1] is its ability to provide accurate Failure Analysis and Failure Forecasts with extremely small samples. With Weibull, solutions are possible at the earliest indications of a problem without having to pursue further. Small samples also allow cost-effective component testing. Maximum Likelihood Estimation (MLE) has been the most widely used method for estimating the parameters of the Constant Shape Bi-Weibull distribution. Recently, Bayesian Estimation approach has received great attention by most researchers among them is [4]. They considered Bayesian Survival Estimator for Weibull distribution with censored data. While [2] studied Bayesian Estimation for the extreme value distribution using progressive censored data and Asymmetric Loss. Bayes Estimator for Exponential distribution with Extension of Jeffreys" Prior Information was considered by [5]. Others including [3, 6, and 12] did some comparative studies on the estimation of Weibull parameters using complete and censored samples and [11] determined Bayes Estimation of the extreme-value Reliability function.

In recent, work we developed Functional Relationship between Brier Score and Area Under the Constant Shape Bi-Weibull ROC Curve [10], Confidence Intervals Estimation for ROC Curve, AUC and Brier Score under the Constant Shape Bi-Weibull Distribution [7], Asymmetric and Symmetric Properties of Constant Shape Bi-Weibull ROC Curve Described by Kullback-Leibler Divergences [8], and Bayesian Estimation of Parameters under the Constant Shape Bi-Weibull Distribution Using Extension of Jeffreys" Prior Information with Three Loss Functions[9].

In this paper, the Bayesian Estimation of Failure Rate under the Constant Shape Bi-Weibull Distribution is studied by Using Extension of Jeffreys" Prior Information with Three Loss Functions. This paper is organized as follows: In Section 2, estimation of Failure Rate under MLE is obtained. In Section 3, Extension of Jeffreys" Prior Information with Three Loss functions is discussed. Section 4, provides simulation study for proposed theory. In Section 5, the proposed theory is validated by using real data. Finally in Section 6 we provide all the findings.

2. Maximum Likelihood Estimation of the Failure Rate for Constant Shape Bi-Weibull Distribution

Let $t_1, t_2 \ldots t_n$ be a random sample of size $\boldsymbol{n}$ with respect to the Constant Shape Bi-Weibull distribution, with $\boldsymbol{\sigma}$ and $\boldsymbol{\beta}$ as the parameters, where **σ** is the scale parameter and **β** is the shape parameter. The probability density function (pdf), cumulative distribution function (cdf) and Failure Rate are given, respectively, as

$$f(t;\sigma,\beta) = \frac{\beta}{\sigma} t^{\beta-1} e^{-\left[\frac{t^{\beta}}{\sigma}\right]}. \tag{1}$$

The Cumulative distribution function (CDF) is

$$F(t;\sigma,\beta) = 1 - e^{-\left[\frac{t^{\beta}}{\sigma}\right]}. \tag{2}$$

The Failure rate is

$$FR(t) = \frac{\beta}{\sigma} t^{\beta-1} . \tag{3}$$

The likelihood function of the pdf is

$$L(t_i,\sigma,\beta) = \prod_{i=1}^{n} \frac{\beta}{\sigma} {t_i}^{\beta-1} e^{-\left[\frac{{t_i}^{\beta}}{\sigma}\right]}. \tag{4}$$

The log-likelihood function is

$$lnL = nln\beta + (\beta-1)\left[\sum_{i=1}^{m} lnt_i\right] - nln\sigma - \frac{1}{\sigma}\sum_{i=1}^{n} {t_i}^{\beta}. \tag{5}$$

By differentiating the equation (5) with respect to σ and **β** and equating to zero, we get

$$\frac{\partial lnL}{\partial \sigma} = -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n} {t_i}^{\beta}}{\sigma^2} = 0 . \tag{6}$$

$$\frac{\partial lnL}{\partial \beta} = \frac{n}{\beta} + \left[\sum_{i=1}^{n} lnt_i\right] - \frac{1}{\sigma}\sum_{i=1}^{n} {t_i}^{\beta}\, lnt_i = 0 . \tag{7}$$

From equation (6), we get

$$\hat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} {t_i}^{\beta} . \tag{8}$$

First we shall find $\hat{\boldsymbol{\beta}}$ and so that $\hat{\boldsymbol{\sigma}}$ can be determined. So that we propose to find $\hat{\boldsymbol{\beta}}$ by using Newton-Raphson method as given below. Let $f(\boldsymbol{\beta})$ be the same as equation (6) and taking the first differential of $f(\boldsymbol{\beta})$, we have

$$f'(\beta) = -\left(\frac{n}{\beta^2}\right) - \frac{1}{\sigma}\sum_{i=1}^{n} {t_i}^{\beta} (lnt_i)^2 . \tag{9}$$

By substituting equation (8) into equation (7), we call $f(\boldsymbol{\beta})$ as

$$f(\beta) = \frac{n}{\beta} + \left[\sum_{i=1}^{n} lnt_i\right] - \frac{\sum_{i=1}^{n} {t_i}^{\beta}\, lnt_i}{\frac{1}{n}\sum_{i=1}^{n} {t_i}^{\beta}} . \tag{10}$$

Substituting equation (8) into equation (9), we obtain

$$f'(\beta) = -\left\{\frac{n}{\beta^2} + \frac{\sum_{i=1}^{n} {t_i}^{\beta} (lnt_i)^2}{\frac{1}{n}\sum_{i=1}^{n} {t_i}^{\beta}}\right\} . \tag{11}$$

Therefore, $\hat{\boldsymbol{\beta}}$ is obtained from the equation below by carefully choosing an initial value $\boldsymbol{\beta}$ as $\boldsymbol{\beta_i}$ and iterating the process till it converges:

$$\beta_{i+1} = \beta_i - \frac{\frac{n}{\beta} + \left[\sum_{i=1}^{n} lnt_i\right] - \frac{\sum_{i=1}^{n} {t_i}^{\beta}\, lnt_i}{\frac{1}{n}\sum_{i=1}^{n} {t_i}^{\beta}}}{-\left\{\frac{n}{\beta^2} + \frac{\sum_{i=1}^{n} {t_i}^{\beta} (lnt_i)^2}{\frac{1}{n}\sum_{i=1}^{n} {t_i}^{\beta}}\right\}} . \tag{12}$$

The estimate of the Failure Rate of the Constant Shape Bi-Weibull distribution under MLE is

$$\widehat{FR}(t) = \frac{\hat{\beta}}{\hat{\sigma}} t^{\hat{\beta}-1} . \tag{13}$$

3. Bayesian Estimation of the Failure Rate for Constant Shape Bi-Weibull Distribution

Bayesian Estimation approach has received a lot of attention in recent times for analyzing Failure time data, which has mostly been proposed as an alternative to that of the traditional methods.

Bayesian Estimation approach makes use of once prior knowledge about the parameters as well as the available data. When once prior knowledge about the parameter is not available, it is possible to make use of the noninformative prior in Bayesian analysis.

Since we have no knowledge on the parameters, we seek to use the Extension of Jeffreys‟ Prior Information, where Jeffreys‟ Prior is the square root of the determinant of the Fisher information.

According to [5], the Extension of Jeffreys‟ prior is obtained by taking $u(\theta) \propto [I(\theta)]^c$, $c \epsilon R^+$, so that

$$u(\theta) \propto \left[\frac{1}{\theta}\right]^{2c}.$$

Thus,

$$u(\sigma,\beta) \propto \left[\frac{1}{\sigma\beta}\right]^{2c}.$$

Given a sample t= (t_1, t_2, ..., t_n) from the likelihood function of the pdf (1) is

$$L(t_i \mid \sigma,\beta) = \prod_{i=1}^{n} \frac{\beta}{\sigma} {t_i}^{\beta-1} e^{-\left[\frac{{t_i}^{\beta}}{\sigma}\right]}.$$

With Bayes theorem, the joint posterior distribution of the parameters $\boldsymbol{\sigma}$ and **β** is

$$\pi^*(\sigma,\beta|t) \propto L(t|\,\sigma,\beta)u(\sigma,\beta)$$

$$L(t_i \mid \sigma,\beta) = \frac{k}{(\sigma\beta)^{2c}} \prod_{i=1}^{n} \frac{\beta}{\sigma} {t_i}^{\beta-1} e^{-\left[\frac{{t_i}^{\beta}}{\sigma}\right]},$$

where $\boldsymbol{k}$ is the normalizing constant that makes $\boldsymbol{\pi}^*$ a proper pdf.

Remark 3.1

Here we consider two Asymmetric Loss Functions namely Linear Exponential Loss Function (LINEX), General Entropy Loss Function and the one Symmetric Loss Function is the Squared Error Loss.

3.1 Linear Exponential (LINEX) Loss Function

The LINEX Loss Function is under the assumption that the minimal loss occurs at $\hat{\theta} = \theta$ and is expressed as

$$L(\hat{\theta} - \theta) \propto \exp\left(a(\hat{\theta} - \theta)\right) - a(\hat{\theta} - \theta) - 1 ,$$

where $\hat{\boldsymbol{\theta}}$ is an estimation of $\boldsymbol{\theta}$ and $a \neq 0$. The sign and magnitude of the shape parameter „**a**" represents the direction and degree of symmetry, respectively. There is overestimation if $\mathbf{a} > 0$ and underestimation if $\mathbf{a} < 0$ but when $\mathbf{a} \cong 0$, the LINEX Loss Function is approximately the Squared Error Loss Function.

The posterior expectation of the LINEX Loss Function, according to [10], is

$$E_\theta L(\hat{\theta} - \theta) \propto exp(a\hat{\theta}) E_\theta(\exp(-a\theta))$$

$$-a(\hat{\theta} - E_\theta(\theta)) - 1 . \qquad (14)$$

The Bayes Estimator of $\boldsymbol{\theta}$, represented by $\hat{\boldsymbol{\theta}}_{BL}$ under LINEX Loss Function, is the value of $\hat{\boldsymbol{\theta}}$ which minimizes equation (14) and is given as

$$\hat{\theta}_{BL} = -\frac{1}{a} ln\, E_\theta(exp(-a\theta)).$$

Provided $E_\theta(exp(-a\theta)$ exists and is finite.

The posterior density function of the Failure Rate under LINEX loss is given as

$$\widehat{FR}(t)_{BL} = E\left\{exp\left(-a\frac{\beta}{\sigma}t_i{}^{\beta-1}\right)|t_i\right\}$$

$$= \frac{\iint exp\left(-a\frac{\beta}{\sigma}t_i{}^{\beta-1}\right)\pi^*(\sigma,\beta)\, d\sigma d\beta}{\iint \pi^*(\sigma,\beta) d\sigma d\beta} . \qquad (15)$$

From (15), it can be observed that ratio of integrals which cannot be solved analytically and for that we employ Lindley‟s approximation procedure to estimate the Failure Rate.

Lindley considered an approximation for the ratio of integrals for evaluating the posterior expectation of an arbitrary function $\hat{\boldsymbol{u}}(\boldsymbol{\theta})$ as

$$E[u(\theta)|x] = \frac{\int u(\theta)v(\theta)[L(\theta)]d\theta}{\int v(\theta)[L(\theta)]d\theta}$$

According to [13], Lindley‟s expansion can be approximated asymptotically by

$$\hat{\theta} = u + \frac{1}{2}[u_{11}\delta_{11} + u_{22}\delta_{22}] + u_1\rho_1\delta_{11} + u_2\rho_2\delta_{22}$$

$$+\frac{1}{2}[L_{30}u_1\delta^2{}_{11} + L_{03}u_2\delta^2{}_{22}] , \qquad (16)$$

where L is the log-likelihood function in equation (5),

$$u = exp\left(-a\frac{\beta}{\sigma}t_i{}^{\beta-1}\right),$$

$$p = \frac{\beta}{\sigma}t_i{}^{\beta-1},$$

$$u_1 = \frac{\partial u}{\partial \sigma} = \frac{a}{\sigma}pu,$$

$$u_{11} = \frac{\partial^2 u}{\partial^2 \sigma} = uq^2\left(\frac{a}{\sigma}\right)^2 - \frac{2auq}{\sigma^2},$$

$$u_2 = \frac{\partial u}{\partial \beta} = -auq\left(lnt_i + \frac{1}{\beta}\right),$$

$$u_{22} = \frac{\partial^2 u}{\partial^2 \beta} = -auq\left[(lnt_i)^2 + \frac{2lnt_i}{\beta}\right]$$

$$+ (aq)^2 u\left[(lnt_i)^2 + \frac{2lnt_i}{\beta} + \frac{1}{\beta^2}\right]$$

$$\rho(\sigma,\beta) = -ln(\sigma^{2c}) - ln(\beta^{2c}),$$

$$\rho_1 = \frac{\partial \rho}{\partial \sigma} = -\frac{1}{\sigma^{2c}},$$

$$\rho_2 = \frac{\partial \rho}{\partial \beta} = -\frac{1}{\beta^{2c}},$$

$$\delta_{11} = (-L_{20})^{-1},\ \delta_{22} = (-L_{02})^{-1},$$

$$L_{02} = -\left(\frac{n}{\beta^2}\right) - \frac{1}{\sigma}\sum_{i=1}^{n} t_i{}^\beta\, (lnt_i\,)^2,$$

$$L_{03} = 2\left(\frac{n}{\beta^3}\right) - \frac{1}{\sigma}\sum_{i=1}^{n} t_i{}^\beta\, (lnt_i\,)^3,$$

$$L_{20} = \frac{n}{\sigma^2} - 2\frac{\sum_{i=1}^{n} t_i{}^\beta}{\sigma^3},$$

$$and\ L_{30} = -2\frac{n}{\sigma^3} + 6\frac{\sum_{i=1}^{n} t_i{}^\beta}{\sigma^4}.$$

3.2 General Entropy Loss Function

Another useful Asymmetric Loss Function is the General Entropy (GE) Loss which is a generalization of the Entropy Loss and is given as

$$L(\hat{\theta} - \theta) \propto \left(\frac{\hat{\theta}}{\theta}\right)^k - k\, ln\left(\frac{\hat{\theta}}{\theta}\right) - 1$$

The Bayes Estimator $\hat{\boldsymbol{\theta}}_{BG}$ of $\boldsymbol{\theta}$ under the General Entropy Loss is

$$\hat{\theta}_{BG} = [\,E_\theta(\theta^{-k})]^{-\frac{1}{k}} \quad ,$$

provided $E_\theta(\theta^{-k})$ exists and is finite.

The posterior density function of the Failure Rate under General Entropy loss is given as

$$\widehat{FR}(t)_{BG} = E\left\{\left(\frac{\beta}{\sigma}t_i{}^{\beta-1}\right)^{-k} |t_i\right\}$$

$$= \frac{\iint \left(\frac{\beta}{\sigma} t_i^{\beta-1}\right)^{-k} \pi^*(\sigma,\beta) d\sigma d\beta}{\iint \pi^*(\sigma,\beta) d\sigma d\beta}.$$

Applying the same Lindley approach here as in (16) with u_1, u_{11} and u_2, u_{22} are the first and second derivatives for $\boldsymbol{\sigma}$ and $\boldsymbol{\beta}$, respectively, and are given as

$$u = \left(\frac{\beta}{\sigma} t_i^{\beta-1}\right)^{-k},$$

$$r = t_i^{\beta-1},$$

$$u_1 = \frac{\partial u}{\partial \sigma} = \frac{k}{\sigma} u,$$

$$u_{11} = \frac{\partial^2 u}{\partial^2 \sigma} = \frac{ku}{\sigma^2}[k-1],$$

$$u_2 = \frac{\partial u}{\partial \beta} = -ku\left(\frac{lnt_i}{\sigma} + \frac{lnt_i}{\beta}\right),$$

$$and\ u_{22} = \frac{\partial^2 u}{\partial^2 \beta} = u\left[\left(\frac{k}{\sigma}\right)^2 (lnt_i)^2 + \frac{2k^2 lnt_i}{\sigma\beta} + \left(\frac{k}{\beta}\right)^2 + \frac{k}{\beta^2}\right].$$

3.3 Symmetric Loss Function

The Symmetric Loss Function is the Squared Error Loss is given by

$$L(\hat{\theta} - \theta) \propto (\hat{\theta} - \theta)^2.$$

This Loss Function is symmetric in nature, that is, it gives equal weightage to both over and under estimation. In real life, we encounter many situations where overestimation may be more serious than underestimation or vice versa.

The most common loss function used for Bayesian estimation is the squared error (SE), also called quadratic loss. The square error loss denotes the punishment in using to $\hat{\theta}$ estimate θ and is given as

$$E_\theta(t|\theta) = (\hat{\theta}(t) - \theta)^2,$$

where the expectation is taken over the joint distribution of θ and (t).

The posterior density function of the Failure Rate under the Symmetric loss function are given as

$$\widehat{FR}(t)_{BS} = E\left\{\frac{\beta}{\sigma} t_i^{\beta-1} | t_i\right\} = \frac{\iint \frac{\beta}{\sigma} t_i^{\beta-1} \pi^*(\sigma,\beta) d\sigma d\beta}{\iint \pi^*(\sigma,\beta) d\sigma d\beta}$$

Applying the same Lindley approach here as in (16) with u_1, u_{11} and u_2, u_{22} are the first and second derivatives for $\boldsymbol{\sigma}$ and $\boldsymbol{\beta}$, respectively, and are given as

$$u = \frac{\beta}{\sigma} t_i^{\beta-1}, d = lnt_i,$$

$$u_1 = \frac{\partial u}{\partial \sigma} = \frac{-u}{\sigma},$$

$$u_{11} = \frac{\partial^2 u}{\partial^2 \sigma} = \frac{2u}{\sigma^2},$$

$$u_2 = \frac{\partial u}{\partial \beta} = u\left(d + \frac{1}{\beta}\right),$$

$$and\ u_{22} = \frac{\partial^2 u}{\partial^2 \beta} = u\left[(d)^2 + \frac{2d}{\beta}\right].$$

4. Simulation Study

Since it is difficult to compare the performance of the estimators theoretically and also to validate the data employed in this paper, we have performed extensive simulations to compare the estimators through Mean Squared Errors and Absolute Biases by employing different sample sizes with different parameter values.

The Mean Squared Error and Absolute Bias given as

$$MSE = \frac{\sum_{r=1}^{5000}(\hat{\theta}^r - \theta)^2}{R-1},$$

$$and\ Abs = \frac{\sum_{r=1}^{5000}|\hat{\theta}^r - \theta|}{R-1}.$$

In our Simulation study, we chose a sample size of $\boldsymbol{n}$= 25, 50, and 100 to represent small, medium, and large dataset. The Failure Rate is estimated for Constant Shape Bi-Weibull distribution with Maximum Likelihood and Bayesian using Extension of Jeffreys‟ Prior methods.

The values of the parameters chosen are $\boldsymbol{\sigma} = 0.5$ and 1.5, $\boldsymbol{\beta} = 0.8$ and 1.2. The values of Jeffreys Extension are $\boldsymbol{c} = 0.4$ and 1.4. The values for the Loss parameters (a, k) are $a = k$ *and* . These were iterated *(R)* 5000 times and the Failure Rate for each method was calculated.

The results are presented below for the estimated Failure Rate and their corresponding Mean Squared Error and Absolute Bias values.

In Table 4.1 we present the Mean Square Error estimated values for the Failure Rate $FR(t)$ for both the MLE and Bayesian Estimation using extension of Jeffrey‟s prior information with the three loss functions.

Table 4.1: MSE Estimated Failure Rate

n	σ	c	β	$\hat{F}(t)_{ML}$	$\hat{F}(t)_{BS}$	$\widehat{FR}(t)_{BL}$	$\widehat{FR}(t)_{BG}$	$\widehat{FR}(t)_{BL}$	$\widehat{FR}(t)_{BG}$	$\widehat{FR}(t)_{BL}$	$\widehat{FR}(t)_{BG}$	$\widehat{FR}(t)_{BL}$	$\widehat{FR}(t)_{BG}$
						a = k = 0.6		a= k = -0.6		a =k =1.6		a = k = -1.6	
25	0.5	0.4	0.8	1.2e+82	63.952	0.5234	0.7623	8728.4	9.2973	0.0579	1.5280	216369	3191.36
	0.5	0.4	1.2	1.0e+37	6.7003	0.8827	0.7005	5.4806	1.8521	0.8998	2.7323	4328.0	43.8584
	0.5	1.4	0.8	2.8e+94	88.072	0.4813	0.8740	85352	11.222	0.0613	1.8849	2.0e+13	10068.2
	0.5	1.4	1.2	1.9e+89	313.17	15.085	28.353	1466.7	74.310	10.339	175.68	627285	1622.32
	1.5	0.4	0.8	5.52822	1.2129	0.1472	2.3995	1.6475	0.5675	0.2056	77.606	113.880	1.63443
	1.5	0.4	1.2	142720	1.1039	0.1998	0.9113	0.5458	0.4718	0.6805	13.159	11.2061	2.33524
	1.5	1.4	0.8	985157	1.5281	0.1982	3.5763	2.0968	0.8470	0.2970	111.63	137.586	2.38728
	1.5	1.4	1.2	537816	2.2130	0.4978	7.4095	0.8755	1.2176	3.1650	517.86	16.4905	3.89148
50	0.5	0.4	0.8	4.e+142	71.561	0.6524	1.1738	9043.2	10.884	0.0827	2.2894	2756106	1556.70
	0.5	0.4	1.2	4.1e+95	7.7536	1.3157	0.6745	6.9430	1.9743	1.6610	2.4988	6847.24	44.7894
	0.5	1.4	0.8	3.e+151	66.066	0.7285	1.2189	4810.3	10.314	0.0919	2.2433	2908489	1055.21
	0.5	1.4	1.2	1.e+172	15.729	3.4073	2.3162	22.240	3.8102	7.2042	15.841	18757.7	63.9147
	1.5	0.4	0.8	2.1e+26	3.3482	0.3807	5.4146	4.7239	1.5286	0.5278	146.75	409.003	5.26739
	1.5	0.4	1.2	33.1864	2.3936	0.5006	2.3669	1.1966	1.0863	1.8681	37.871	23.8561	4.83172
	1.5	1.4	0.8	2.2e+15	4.7178	0.4817	5.2441	7.8232	1.7973	0.5945	132.66	1169.90	8.54443
	1.5	1.4	1.2	57.5080	1.9741	0.4068	1.7615	0.9162	0.8784	1.4526	25.286	14.8790	4.02218
100	0.5	0.4	0.8	1.e+213	114.46	1.2803	2.5970	4618.2	19.254	0.1622	4.7920	4004063	1126.63
	0.5	0.4	1.2	1.e+113	29.440	6.6789	3.4804	16.983	6.7853	17.906	31.256	11183.0	141.765
	0.5	1.4	0.8	0.e+312	418.70	1.7447	2.9458	590378	45.327	0.1988	4.8621	4.7e+20	25821.2
	0.5	1.4	1.2	1.e+258	28.507	6.2988	2.7501	23.452	6.5210	13.801	14.633	19213.2	135.293
	1.5	0.4	0.8	2.5e+38	5.7759	0.6330	9.0292	7.8450	2.7303	0.8209	216.73	583.030	9.04485
	1.5	0.4	1.2	63.3259	4.9776	1.0989	4.8070	2.6086	2.2283	4.0082	78.432	46.1261	9.74658
	1.5	1.4	0.8	6.2e+42	10.637	1.0995	11.657	16.431	4.3427	1.3881	266.77	1962.18	20.4616
	1.5	1.4	1.2	1.2e+26	4.9257	1.1539	5.7377	2.5014	2.3034	4.7023	104.78	45.2562	9.17522
ML: Maximum Likelihood, BS: Squared Error Loss function, BL: LINEX Loss function, BG: General Entropy Loss function.													

From Table 4.1 it is observed that Bayes estimation with LINEX loss function provides the smallest MSE values in most cases especially when the loss parameter values are 0.6 and 1.6. Also sample size increases MLE and Bayes estimation under all loss functions have increases in MSE values.

In Table 4.2 we present the Absolute Bias estimated values for the Failure Rate $FR(t)$ for both the Maximum Likelihood Estimation and Bayesian Estimation using extension of Jeffreys" prior information with the three loss functions.

Table 4.2: Absolute Bias Estimated Failure Rate

n	σ	c	β	$\widehat{FR}(t)_{ML}$	$\widehat{FR}(t)_{BS}$	$\widehat{FR}(t)_{BL}$	$\widehat{FR}(t)_{BG}$	$\widehat{FR}(t)_{BL}$	$\widehat{FR}(t)_{BG}$	$\widehat{FR}(t)_{BL}$	$\widehat{FR}(t)_{BG}$	$\widehat{FR}(t)_{BL}$	$\widehat{FR}(t)_{BG}$
						a = k = 0.6		a= k = -0.6		a =k =1.6		a = k = -1.6	
25	0.5	0.4	0.8	1.1e+41	26.281	2.6640	3.3840	226.04	10.3654	0.9064	4.9076	242299.4	129.697
	0.5	0.4	1.2	3.2e+18	10.541	3.6603	3.1271	9.3734	5.57900	3.3612	5.4761	270.3620	26.7775
	0.5	1.4	0.8	1.6e+47	25.668	2.5538	3.7212	458.36	10.4216	0.9752	5.5539	6263878	157.151
	0.5	1.4	1.2	4.3e+44	87.762	18.517	24.687	182.99	42.8530	11.606	46.796	3535.722	194.600
	1.5	0.4	0.8	9.69623	3.7197	1.3377	6.0348	4.2632	2.68147	1.4948	34.886	31.97229	4.04182
	1.5	0.4	1.2	7734.58	4.0614	1.6072	3.2992	2.9231	2.56395	2.6745	11.651	13.52524	6.03844
	1.5	1.4	0.8	109920	4.8660	1.7579	7.7213	5.7159	3.59762	2.1373	42.354	44.9636	5.57046
	1.5	1.4	1.2	735570	5.7236	2.5545	7.6941	3.6471	4.02288	5.5168	48.034	14.9602	7.85227
50	0.5	0.4	0.8	2.0e+71	39.851	4.1845	6.0704	276.82	16.3766	1.4530	8.4473	250738.8	135.417
	0.5	0.4	1.2	6.3e+47	15.658	6.2065	4.3346	15.821	7.9881	6.0480	7.2957	472.0624	37.6948
	0.5	1.4	0.8	6.2e+75	39.6262	4.4709	6.1585	256.78	16.3923	1.6418	8.4781	1125170	132.043
	0.5	1.4	1.2	1.3e+86	22.84	10.022	7.7003	31.140	11.3396	12.633	16.440	783.7711	47.2651
	1.5	0.4	0.8	1.4e+13	9.2811	3.1634	3.1634	10.822	6.54346	3.5870	68.498	89.42168	10.7531
	1.5	0.4	1.2	39.9529	8.6945	3.9492	8.3192	6.0083	5.84233	7.3531	31.472	23.88736	12.3097
	1.5	1.4	0.8	468580	9.8109	3.2506	12.730	11.815	6.60159	3.4071	65.184	109.9035	11.6327
	1.5	1.4	1.2	50.7945	8.0100	3.6315	7.4523	5.3680	5.34833	6.7166	27.414	19.63148	11.3913
100	0.5	0.4	0.8	3.e+106	78.246	8.5928	12.926	413.63	32.9757	2.9381	17.702	79622.86	223.5619
	0.5	0.4	1.2	8.e+223	42.260	18.551	11.903	28.604	20.1363	24.078	24.790	798.6049	5.4806
	0.5	1.4	0.8	1.e+267	108.54	9.5928	13.466	11401	41.4398	3.2555	17.352	30854204	467.214
	0.5	1.4	1.2	2.e+188	42.829	19.535	12.190	40.132	20.5038	24.948	23.464	1089.806	93.2288
	1.5	0.4	0.8	1.5e+19	17.411	5.7708	23.864	20.005	12.3722	6.1474	118.19	159.5608	20.6433
	1.5	0.4	1.2	76.9857	17.999	8.3498	16.690	12.962	11.9500	15.334	61.984	51.16199	25.3043
	1.5	1.4	0.8	2.5e+21	21.631	7.0684	26.897	25.609	14.7635	7.4192	130.91	232.0705	27.0839
	1.5	1.4	1.2	1.1e+13	17.493	8.3340	17.553	12.337	11.8239	16.040	69.217	47.90307	24.0546
ML: Maximum Likelihood, BS: Squared Error Loss function, BL: LINEX Loss function, BG: General Entropy Loss function.													

From Table 4.2 it is observed that Bayes estimation with LINEX loss function provides the smallest Absolute Bias values in most cases especially when the loss parameter values are (0.6, 1.6).

As the sample size increases Absolute values of the MLE and Bayes estimation under all loss functions increases.

5. Illustration

The real data set is about a clinical Trial in the Treatment of Carcinoma of the Oropharynx (PHARYNX) Data extracted from [16].

The data file gives the data for a part of a large clinical trial carried out by the Radiation Therapy Oncology Group in the United States.

This data consists of a total of 195 respondents of which 53 are alive and 142 are dead. Here we considered age in years at time of diagnosis is the most factors. Table 5.1 depicts the Standard Error values for Estimated Failure Rate $FR(t)$ using PHARYNX Data.

Table 5.1: Standard Error values for Estimated Failure Rate $FR(t)$ Using PHARYNX Data

Estimates	MLE	BL	BG	BS
FR(t)	3.286e-05	1.652e-05	0.00093	0.00332

Here ML: Maximum Likelihood,
BS: Squared Error Loss function,
BL: LINEX Loss function, and
BG: General Entropy Loss function.

From Table 5.1, we observe that, Bayesian estimator under LINEX loss function has the smallest value 1.652e-05 for Failure Rate $FR(t)$.

So that the Bayes estimators of Failure Rate $FR(t)$ under LINEX loss function is best estimation method for Constant Shape Bi-Weibull Distribution using PHARYNX Data.

6. Conclusion

In this paper, we have addressed the problem of Bayesian estimation of Failure Rate for the Constant Shape Bi-Weibull distribution, under three Loss functions, namely, the Linear Exponential (LINEX) Loss, General Entropy Loss, and Square Error Loss functions and that of Maximum Likelihood Estimation.

Bayes estimators were obtained using Lindley approximation while MLE were obtained using Newton-Raphson method.

A Simulation study was conducted to examine and compare the performance of the estimates for different sample sizes with different values for the extension of Jeffreys" prior and the loss functions.

From the results, we observe that in most cases, Bayesian estimator under LINEX loss function has the smallest Mean Squared Error values and minimum Bias for Failure Rate $FR(t)$ in most cases especially when the loss parameter values are (0.6, 1.6), for both values of the extension of Jeffreys" prior information.

As the sample size increases the Mean Squared Error and the Absolute Bias for Maximum Likelihood Estimator and Bayes estimator under all the loss functions increases correspondingly.

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Author Profile

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