Some New Results about the Convergence of Fuzzy Measurable Functions Sequence on Fuzzy Measure on Fuzzy Sets

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Abstract: The goal of this paper is to study the Convergence of fuzzy measurable functions sequence on fuzzy sets and get on some new results.

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1. Introduction

In measure theory, several types of convergence were introduced for sequence of measurable functions on a measure space and some basic relations among these types were established [3].

In the proof of the theorem (9) we need thatμ is countably weakly null-additive because as is well known, sugeno's fuzzy measure loses additivity in general therefore if μ(An) = 0 for all

\[ n \geq 1 \Rightarrow \mu \left( \bigcup_{n=1}^{\infty} A_n \right) = 0, \{A_n\} \subset \mathcal{F} \]

Fuzzy measure generalization of measure theory. This generalization is obtained by replacing the additivity axiom of measure theory with weak axiom of monotonicity and continuity [1].

The fuzzy measure, defined on \( \sigma \)-field, was introduced by Sugeno [20]. Ralescu and Adams [21] generalized the concepts of fuzzy measure and fuzzy integral to the case that the value of a fuzzy measure can be infinite, and to realize an approach from Subjective.

Jun Li [4] study order continuous and strongly order continuous of monotone set function and convergence of measurable functions sequence

Jun Li, Masami Yasuda, Qingshan Jiang, Hisakichi Suzuki and Zhenyuan Wang [2], Deli Zhang and Caimei Guo [5], studied some Convergence of sequence of measurable functions on Fuzzy measure spaces and generalized convergence theorems obtained a series of new results.

After that, many authors studied Convergence of sequence of measurable functions on Fuzzy measure spaces and proved some results about it as G. J. Kirl [6, 7], Jun Li, Radko Mesiar, Endre Pap and Erich Peter Klement [8], L.Y. Kui[9], L.Y. Kui and L. Baoding [10].

In this paper, we mention the definition of Fuzzy Measure on Fuzzy Set and study three types of convergence of sequence of fuzzy measurable functions defined on fuzzy sets; the concepts of "almost" and "Pseudo" are introduced also to the Convergence almost everywhere, Convergence in fuzzy measure and almost uniformly convergence and get on some new results about them.

Definition (1): [17, 18]
Let \( \Omega \) be an empty set, a fuzzy set \( A \) in \( \Omega \) (ora fuzzy subset inf\( \Omega \)) is a function from \( \Omega \) into \( I \), i.e. \( A \in I^\Omega \). \( A(\cdot) \) is interpreted as the degree of membership of element \( x \) in a fuzzy set \( A \) for each \( x \in \Omega \), a fuzzy set \( A \) in \( \Omega \) is can be represented by the set of pairs:

\[ A = \{(x, A(x)) : x \in \Omega \} \]

Note that every ordinary set is fuzzy set, i.e. \( P(\Omega) \subseteq I^\Omega \).

Definition (2): [11, 12]
A family \( \mathcal{F} \) of fuzzy sets in a set \( \Omega \) is called a fuzzy \( \sigma - \)field on a set \( \Omega \) if,

1) \( \phi, \Omega \in \mathcal{F} \).
2) If \( A \in \mathcal{F} \), then \( A^c \in \mathcal{F} \).
3) If \( \{A_n\} \subset \mathcal{F}, n = 1, 2, 3, \ldots \), then \( \bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \).

Evidently, an arbitrary \( \sigma - \)field must be fuzzy \( \sigma - \)field.

A fuzzy measurable Space is a pair \((\Omega, \mathcal{F})\), where \( \Omega \) is a set and \( \mathcal{F} \) is a fuzzy \( \sigma - \)field on \( \Omega \), a fuzzy set \( A \) in \( \Omega \) is called fuzzy measurable (fuzzy measurable with respect to the fuzzy \( \sigma - \)field) if \( A \in \mathcal{F} \), i.e. any member of \( \mathcal{F} \) is called a fuzzy measurable set.

Definition (3): [13]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \( \mu : \mathcal{F} \rightarrow [0, \infty] \) is said to be a fuzzy measure on \((\Omega, \mathcal{F})\) if it satisfies the following properties:

1) \( \mu(\emptyset) = 0 \)
2) If \( A, B \in \mathcal{F} \) and \( A \subseteq B \), then \( \mu(A) \leq \mu(B) \)

Definition (4): [14]
Let \((\Omega, \mathcal{F})\) be a fuzzy measurable space. A set function \( \mu : \mathcal{F} \rightarrow [0, \infty] \) is said to be

1. Exhaustive if \( \mu(A_n) \rightarrow 0 \) whenever \( \{A_n\} \) is infinite sequence of disjoint sets in \( \mathcal{F} \)
2. Order-continuous if $\mu(A_n) \to 0$, whenever $A_n \in \mathcal{F}$, $n = 1, 2, ...$ and $A_n \downarrow \emptyset$.

**Definition (5): [14, 15]**
Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \to [0, \infty)$ is said to be Null-additive, if $\mu(A \cup B) = \mu(A)$ whenever $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$, and $\mu(B) = 0$.

**Definition (6): [1]**
Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \to [0, \infty)$ is said to be weakly null-additive, if for any $A, B \in \mathcal{F}$,

$$\mu(A) = \mu(B) = 0 \implies \mu(A \cup B) = 0$$

**Remark (70):**
The concept of null-null additive stems from a wings textbook which the book[1] derived from, in which it is said to be weak null additive. But we consider that it is more precise and vivid to call it "null-null additive"

**Definition (8): [16]**
Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \to [0, \infty)$ is said to be Countably weakly null-additive, if for any $\{A_n\} \subset \mathcal{F}$, $\mu(A_n) = 0$,

$$\text{for all } n \geq 1 \implies \mu \left( \bigcup_{n=1}^{\infty} A_n \right) = 0$$

**Definition (9): [16]**
Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \to [0, \infty)$ is said to be Null-continuous, if $\mu(A_n) = 0$ for every increasing sequence $\{A_n\} \subset \mathcal{F}$ such that $\mu(A_n) = 0$, for all $n \geq 1$.

**Definition (10): [19]**
Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \to [0, \infty)$ is said to be Autocontinuous from above (resp. autocontinuous from below), if $\mu(B_n) \to 0$ implies $\mu(A \cup B_n - \mu(A)) \in \mathcal{F}$ (resp. $\mu(A \cap B_n - \mu(A)) \in \mathcal{F}$), whenever $A \in \mathcal{F}, \{B_n\} \subset \mathcal{F}$, $\mu$ is called autocontinuous if it is both autocontinuous from above and autocontinuous from below.

**Definition (11): [14]**
Let $C(\Omega)$ be the collection of all real valued functions defined on a set $\Omega$. Let $f, f_n \in C(\Omega), n \in \mathbb{N}$ and $A \in \Omega$, we say that

1- $\{f_n\}$ converges pointwise to $f$ on $A$, if for every $x \in A$ and for every $\varepsilon > 0$ there is $k \in \mathbb{N}^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$.

We write $\lim_{n \to \infty} f_n(x) = f(x)$ or $f_n \to f$ on $A$.

2- $\{f_n\}$ uniformly converges to $f$, if for every $\varepsilon > 0$ there is $k \in \mathbb{N}^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > k$ and all $x \in A$.

We write $f_n(x) \to f(x)$ on $A$.

3- $\{f_n\}$ is pointwise Cauchy sequence on $A$, if for every $x \in A$ and for every $\varepsilon > 0$ there is $k \in \mathbb{N}^+$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m > k$, we write $f_n p. c. on A$.

- This has meaning only if $f_n: \Omega \to \mathbb{R}$ is finite valued, because $\mathbb{R}$ is complete it is clear that if $\{f_n\}$ is a Cauchy sequence point wise on $\Omega$, there must be one $f: \Omega \to \mathbb{R}$ such that $f_n \to f$ on $\Omega$.

4- $\{f_n\}$ is a uniformly Cauchy sequence on $A$, if for every $\varepsilon > 0$ there is $k \in \mathbb{N}^+$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m > k$ and all $x \in A$. We write $f_n \ u. c. on A$.

**Definition (12): [1, 19]**
Let $(\Omega, \mathcal{F}, \mu)$ is a fuzzy measure space, let $f, f_n \in C(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, we say that

1) $\{f_n\}$ Converges almost everywhere to $f$ on $A$, denoted by $f_n \to f \ a. e. \ on A$, if there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n \to f$ on $A/B$.

2) $\{f_n\}$ Converges pseudo almost everywhere to $f$ on $A$, denoted by $f_n \to f \ p.a.e. \ on A$, if there is a subset $B \subseteq A$ such that $\mu(A/B) = \mu(A)$ and $f_n \to f$ on $A/B$.

3) $\{f_n\}$ converges almost uniformly to $f$ on $A$, denoted by $f_n \to f \ a.u. \ on A$, if there is a sequence $\{A_n\}$ in $\mathcal{F}$ with $\lim_{n \to \infty} \mu(A_n) = 0$ such that $f_n \to f$ on $A/A_n$ for any fixed $n = 1, 2, ...$

4) $\{f_n\}$ converges pseudo almost uniformly to $f$ on $A$, denoted by $f_n \to f \ p.a.u. \ on A$, if there is a sequence $\{A_n\}$ in $\mathcal{F}$ with $\lim_{n \to \infty} \mu(A/A_n) = \mu(A)$ Such that $f_n \to f$ on $A$ for any fixed $n = 1, 2, ...$

5) $\{f_n\}$ convergence in measure to $f$ on $A$, denoted by $f_n \to f \ m. \ on A$, if $\lim_{n \to \infty} \mu\{x \in A: |f_n(x) - f(x)| \geq \varepsilon\} = 0$ for each $\varepsilon > 0$.

6) $\{f_n\}$ convergence pseudo in measure to $f$ on $A$, denoted by $f_n \to f \ p.m. \ on A$, if $\lim_{n \to \infty} \mu\{x \in A: |f_n(x) - f(x)| < \varepsilon\} = \mu(A)$ for each $\varepsilon > 0$.

**note that**, in the above definitions, when $A = \Omega$ we can omit "on A" from the statements.

2. Main Results

**Lemma (1): [1]**
Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space. if $\mu: \mathcal{F} \to [0, \infty)$ is a non-decreasing set function, then the following statements are equivalent:

1) $\mu$ is null additive
2) $\mu(A \cup B) = \mu(A)$ Whenever $A, B \in \mathcal{F}$ and $\mu(B) = 0$.
3) $\mu(A \cap B) = \mu(A)$ Whenever $A, B \in \mathcal{F}$ such that $\mu(A \cap B) = 0$.
4) $\mu(A \cup \emptyset) = \mu(A)$ Whenever $A \in \mathcal{F}$ and $\mu(\emptyset) = 0$.
5) $\mu(A \cap \emptyset) = \mu(A)$ Whenever $A \in \mathcal{F}$ and $\mu(\emptyset) = 0$.

**Theorem (2):**
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that $\mu$ is null additive, let

$f, f_n \in C(\Omega), n \in \mathbb{N}$ and let $A \in \mathcal{F}$, if $f_n \to f$ on $A$ then $f_n \to f$ on $A/B$.

**Proof:**
Since $f_n \to f$ on $A$, then there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f_n \to f$ on $A/B$. 

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Since µ is null additive, hence
μ(A ∪ B) = μ(A), whenever A, B ∈ ℱ such that A ∩ B = ∅ and μ(B) = 0
By using lemma (1), we get on

\[ μ(A/B) = μ(A ∪ B) = μ(A) \]

Consequently

\[ f_n \to f \text{ on } A. \]

**Theorem (3):**
Let (Ω, ℱ, μ) be a fuzzy measure space such that μ is autocontinuous from below, let \( f, f_n ∈ C(Ω), n ∈ N \) and let A ∈ ℱ, if \( f_n ∈ f \) on A then \( f_n \to f \) on A.

**Proof:**
Since \( f_n \to f \) on A, then there is a subset B ⊆ A such that μ(B) = 0 and \( f_n \to f \) on B/A.

Since μ autocontinuous from below, hence

\[ \lim_{n→∞} μ(A_n) = μ(A), \] whenever
A ∈ ℱ, A_n ∈ ℱ, A_n ⊆ A, n = 1, 2, ... and \( \lim_{n→∞} μ(A_n) = 0 \)

Take B = A_n, n = 1, 2, ..., we have

\[ μ(B) = \lim_{n→∞} μ(A_n) = 0 \]

\[ ⇒ μ(A/B) = \lim_{n→∞} μ(A/A_n) = μ(A) \]

Consequently

\[ f_n \to f \text{ on } A. \]

**Theorem (4):**
Let (Ω, ℱ, μ) be a fuzzy measure space, let \( f, f_n ∈ C(Ω), n ∈ N \) and let A ∈ ℱ if \( f_n \to f \) on A and \( f_n \to f \) on A Then μ is order continuous and autocontinuous from below.

**Proof:**
Since \( f_n \to f \) on A and \( f_n \to f \) on A.

Since \( f_n \to f \) then there is a sequence \( \{A_n\} \) is a sequence of sets in ℱ

With \( \lim_{n→∞} μ(A_n) = 0 \),
i.e. \( μ(A_n) \to 0 \) as n → ∞

Therefore \( A_n ↓ ∅ \)

Consequently μ is order continuous.

To prove μ is autocontinuous from below
Let A ∈ ℱ, \( \{A_n\} \) be a sequence of sets in ℱ with \( A_n ⊆ A \)

Through \( f_n \to f \) on A, we have

\[ \lim_{n→∞} μ(A/A_n) = μ(A) \]

Which is μ is autocontinuous from below.

**Theorem (5):**
Let (Ω, ℱ, μ) be a fuzzy measure space, f, f_n ∈ C(Ω), n ∈ N and let A ∈ ℱ such that \( \lim_{n→∞} μ(A_n) = μ(A) \), whenever \( \{A_n\} \) is a sequence of sets in ℱ with \( \lim_{n→∞} μ(A_n) = 0 \), if f_n \to f on A then f_n \to f on A.

**Proof:**
Since \( f_n \to f \) on A, then there is a sequence \( \{A_n\} \) in ℱ with \( \lim_{n→∞} μ(A_n) = 0 \) such that f_n \to f on A/A_n for any fixed n = 1, 2, ...

Since \( \lim_{n→∞} μ(AΔA_n) = μ(A) \), we have

\[ A \cap A_n ∈ ℱ \] and \( μ(A ∩ A_n) ≤ μ(A_n) \)

So we have

\[ \lim_{n→∞} μ(A ∩ A_n) = 0 \]

and therefore, by the condition given in this theorem, we have

\[ \lim_{n→∞} μ(A/A_n) = μ(A) \]

Consequently

\[ f_n \to f \text{ on } A. \]

**Theorem (6):**
Let (Ω, ℱ, μ) be a fuzzy measure space, f, f_n ∈ C(Ω), n ∈ N and let A ∈ ℱ such that μ is exhaustive, if f_n \to f on A then f_n \to f on A.

**Proof:**
Since f_n \to f on A, then there is a sequence \( \{A_n\} \) of sets in ℱ such that \( \lim_{n→∞} μ(A_n) = μ(A) \) such that f_n \to f on A/A_n for any fixed n = 1, 2, ...

Since μ is exhaustive, let \( \{A_n\} \) is a pairwise of disjoint sequence in ℱ, with \( \lim_{n→∞} μ(A_n) = 0 \)

Consequently

\[ f_n \to f \text{ on } A. \]

**Theorem (7):**
Let (Ω, ℱ, μ) be a fuzzy measure space, f, f_n ∈ C(Ω), n ∈ N and let A ∈ ℱ such that μ is null additive, for any decreasing sequence \( \{A_n\} \) of sets in ℱ, and for any n = 1, 2, ...

Since

\[ A/A_n \uparrow A/(\bigcap_{n=1}^{∞} A_n) \]

and

\[ μ(\bigcap_{n=1}^{∞} A_n) = 0 \]

By using lemma (1) continuity of μ, it follows that

\[ \lim_{n→∞} μ(A/A_n) = \lim_{n→∞} μ(A/(\bigcap_{n=1}^{∞} A_n)) = μ(A) \]

Consequently

\[ f_n \to f \text{ on } A. \]

**Theorem (8):**
Let (Ω, ℱ, μ) be a fuzzy measure space such that μ is weakly null additive, let f, f_n ∈ C(Ω), n ∈ N and let A ∈ ℱ, if f_n \to f on A then \( \lim_{n→∞} μ(f_n) \to f \) on A.

**Proof:**
Since f_n \to f on A, then there is a subset B in ℱ such that μ(B) = 0 and for any x ∈ B \( \lim_{n→∞} μ(f_n(x)) = f(x) \)

For any ε > 0
Since

\[ \{x ∈ Ω: |f_n(x) − f(x)| ≥ ε \} \cap A ≤ B \cup \{x ∈ Ω: |f_n(x) − f(x)| ≥ ε \} \]

By monotonicity and weakly null additive, we have
\( \mu : (x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon) \cap A \rightarrow 0 \) as \( n \rightarrow \infty \)

Consequently \( f_n \rightarrow f \) on \( A \).

**Theorem (9):**

Let \( (\Omega, \mathcal{F}, \mu) \) be a fuzzy measure space such that \( \mu \) is countably weakly null-additive, let \( f, f_n \in \mathcal{C}(\Omega) \), \( n \in \mathbb{N} \) and \( a.e. \) let \( A \in \mathcal{F} \) if \( f_n \rightarrow f \) on \( A \) then

(1) \( f_n \) is a Cauchy a.e.

(2) If \( g \) is real-valued measurable function and \( f_n \rightarrow g \) then \( f = g \) a.e.

(3) If \( g \) is real-valued measurable function such that \( f = g \) a.e. then \( f_n \rightarrow f \).

(4) If \( \{g_n\} \) is a sequence of real-valued measurable functions such that \( f_n = g_n \) a.e. for each \( n \) then \( g_n \rightarrow f \).

(5) If \( g, \{g_n\} \) is a sequence of real-valued measurable function such that \( f_n = g_n \) a.e. for each \( n \) and \( f = g \) a.e. then \( g_n \rightarrow g \).

**Proof:**

(1) Since \( f_n \rightarrow f \) on \( A \), then there is a subset \( B \subseteq A \) such that \( \mu(B) = 0 \) and \( f_n(x) \rightarrow f(x) \) for all \( x \in A \) \( \implies f_n(x) \) is Cauchy sequence for all \( x \in A/B \) \( \implies f_n \) is Cauchy a.e.

(2) Since \( f_n \rightarrow f \) on \( A \), then there is a subset \( B \subseteq A \) such that \( \mu(B) = 0 \) and \( f_n(x) \rightarrow f(x) \) for all \( x \in A/B \)

Since \( f_n \rightarrow g \) on \( A \), then there is a subset \( C \subseteq A \) such that \( \mu(C) = 0 \) and \( f_n(x) \rightarrow g(x) \) for all \( x \in A/C \)

Let \( D = B \cup C \)

(3) Since \( f_n \rightarrow f \) on \( A \), then there is a subset \( B \subseteq A \) such that \( \mu(C) = 0 \) and \( f_n(x) \rightarrow f(x) \) for all \( x \in A/C \)

\( f(x) = g(x) \) for all \( x \in A/C \)

Let \( D = B \cup C \)

(4) Since \( f_n \rightarrow f \) on \( A \), then there is a subset \( B \subseteq A \) such that \( \mu(B) = 0 \) and \( f_n(x) \rightarrow f \) on \( A/B \)

Since \( f_n = g_n \) a.e. then there is a sequence \( \{B_n\} \) such that \( \mu(B_n) = 0 \) and \( f_n(x) = g_n(x) \) for all \( x \notin B_n \)

Let \( D = B \cup C \cup \bigcup_{n=1}^{\infty} B_n \)

\[ \implies \mu(D) = \mu \left( (B \cup C) \cup \bigcup_{n=1}^{\infty} B_n \right) \]

Since \( \mu \) is countably weakly null-additive, we have

\( \mu(D) = 0 \) for all \( x \in D \)

\( \implies \lim_{n \rightarrow \infty} g_n(x) = \lim_{n \rightarrow \infty} f_n(x) = f(x) \) for all \( x \notin D \)

\( \implies g_n(x) \rightarrow f(x) \) for all \( x \notin D \)

Consequently \( g_n \rightarrow f \).

(5) Since \( f_n \rightarrow f \) on \( A \), then there is a subset \( B \subseteq A \) such that \( \mu(B) = 0 \) and \( f_n(x) \rightarrow f(x) \) for all \( x \in B \)

Since \( f_n = g_n \) a.e. for each \( n \) then there is \( B_n \subseteq A \) such that \( \mu(B_n) = 0 \) for all \( n \) and \( f_n(x) = g_n(x) \) for all \( x \in B_n \)

Since \( f = g \) a.e. then there is a subset \( C \subseteq A \) such that \( \mu(C) = 0 \) and \( f(x) = g(x) \) for all \( x \notin C \)

Let

\[ D = B \cup C \cup \bigcup_{n=1}^{\infty} B_n \]

\[ \implies \mu(D) = \mu \left( (B \cup C) \cup \bigcup_{n=1}^{\infty} B_n \right) \]

Since \( \mu \) is countably weakly null-additive, we have

\( \mu(D) = 0 \) for all \( x \in D \)

\( \implies \lim_{n \rightarrow \infty} g_n(x) = \lim_{n \rightarrow \infty} f_n(x) = f(x) \) for all \( x \notin D \)

\( \implies g_n(x) \rightarrow f(x) \) for all \( x \notin D \)

Consequently \( g_n \rightarrow f \).

**Theorem (10):**

Let \( (\Omega, \mathcal{F}, \mu) \) be a fuzzy measure space such that \( \mu \) is countably weakly null-additive, let \( f_n, g, f \in \mathcal{C}(\Omega) \), \( n \in \mathbb{N} \) and let \( A \in \mathcal{F} \) if \( f_n \rightarrow f \) and \( g_n \rightarrow g \) on \( A \) then

(1) \( c. f_n \rightarrow c. f \).

(2) \( f_n + g_n \rightarrow f + g \).

(3) \( |f_n| \rightarrow |f| \).

(4) \( f_n = g_n \) a.e. for all \( n \), then \( f = g \) a.e.

**Proof:**

(1) Since \( f_n \rightarrow f \) , then there is a subset \( B \subseteq A \) such that \( \mu(B) = 0 \) and \( f_n(x) \rightarrow f(x) \) for all \( x \in A/B \) then \( c. f_n(x) \rightarrow c. f(x) \) all \( x \in A/B \)

\[ \implies c. f_n \rightarrow c. f. \]

(2) Since \( f_n \rightarrow f \) on \( A \), then there is a subset \( B \subseteq A \) such that \( \mu(B) = 0 \) and \( f_n(x) \rightarrow f(x) \) for all \( x \in A/B \)

Since \( g_n \rightarrow g \) on \( A \), then there is a subset \( C \subseteq A \) such that \( \mu(C) = 0 \) and \( g_n(x) \rightarrow g(x) \) for all \( x \in A/C \)

Let \( D = B \cup C \)

\[ \implies \mu(D) = \mu(B \cup C) \]

Since \( \mu \) is countably weakly null-additive, we have

\( \mu(D) = 0 \) for any \( x \in A/D \)

\[ \implies \lim_{n \rightarrow \infty} f_n(x) = f(x) = g(x) \]

Consequently \( f_n \rightarrow g \).

(4) Since \( f_n \rightarrow f \) on \( A \), then there is a subset \( B \subseteq A \) such that \( \mu(B) = 0 \) and \( f_n(x) \rightarrow f(x) \) on \( A/B \)

Since \( f_n = g_n \) a.e. then there is a sequence \( \{B_n\} \) such that \( \mu(B_n) = 0 \) and \( f_n(x) = g_n(x) \) for all \( x \notin B_n \)

Let

\[ D = B \cup C \cup \bigcup_{n=1}^{\infty} B_n \]

\[ \implies \mu(D) = \mu \left( (B \cup C) \cup \bigcup_{n=1}^{\infty} B_n \right) \]

Since \( \mu \) is countably weakly null-additive, we have

\( \mu(D) = 0 \) for any \( x \in A/D \)

\[ \implies f_n(x) \rightarrow f(x) \) and \( g_n(x) \rightarrow g(x) \) for all \( x \in D \)

So \( f_n(x) + g_n(x) \rightarrow f(x) + g(x) \), for all \( x \notin D \)

\[ \implies f_n + g_n \rightarrow f + g. \]

(3) Since \( f_n \rightarrow f \) on \( A \), then there is a subset \( B \subseteq A \)
such that $\mu(B) = 0$ and $f_n(x) \to f(x)$ for all $x \in A/B$
\[ \lim_{n \to \infty} f_n(x) = f(x) \text{ for all } x \in A/B \]
\[ \therefore |f_n(x) - f(x)| \to 0 \text{ for all } x \in A/B \]

(4) Since $f_n \xrightarrow{a.e} f$ on $A$, then there is a subset $B \subseteq A$
such that $\mu(B) = 0$ and $f_n(x) \to f(x)$ for all $x \in A/B$

Since $g_n \xrightarrow{a.e} g$ on $A$, then there is a subset $C \subseteq A$ such
that $\mu(C) = 0$ and $g_n(x) \to g(x)$ for all $x \in A/C$

Since $f_n = g_n \text{ a.e.}$ for each $n$ then there is a sequence $\{B_n\}$
such that $\mu(B_n) = 0$ and $f_n(x) = g_n(x)$ for all $x \in B_n$

Let
\[ D = B \cup C \bigcup \left( \bigcup_{n=1}^{\infty} B_n \right) \]
\[ \Rightarrow \mu(D) = \mu \left( B \cup C \bigcup \left( \bigcup_{n=1}^{\infty} B_n \right) \right) \]

Since $\mu$ is countably weakly null-additive, we have
\[ \mu(D) = 0 \text{ for all } x \notin D \]
\[ \therefore f = g \text{ a.e.} \]

Theorem (11):
Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that $\mu$ is
countably weakly null-additive, let $f_n, g_n, f, g \in \mathcal{C}(\Omega), n \in \mathbb{N}$
and let $A \in \mathcal{F}, C \in \mathcal{R}$ if $f_n \xrightarrow{a.e} f$ and

(1) $\|f_n\| \geq g \text{ a.e.}$ then $f \geq g \text{ a.e.}$

(2) $\|f_n\| \geq g \text{ a.e.}$ for each $n$ then $f \leq g$ a.e.

(3) $\|f_n\| \geq |g| \text{ a.e.}$ then $|f| \geq |g|$ a.e.

(4) If $f_n \xrightarrow{a.e} f$ then $f \xrightarrow{a.e.}$

(5) If $f_n \xrightarrow{a.e} f$, $g_n \xrightarrow{a.e} g$ and

$f_n = g_n \text{ a.e.}$ then $f \xrightarrow{a.e.}$

Proof:
Since $f_n \xrightarrow{a.e} f$, then there is $B \subseteq \Omega$ such that $\mu(B) = 0$
and $f_n(x) \to f(x)$ for all $x \in B$

(1) Since $f_n \geq g$ a.e for each $n$, then there is $B_n \subseteq \Omega$
such that $\mu(B_n) = 0$ and $f_n(x) \geq g$ for all $x \in B_n$

Let
\[ D = B \bigcup \left( \bigcup_{n=1}^{\infty} B_n \right) \]
\[ \Rightarrow \mu(D) = \mu \left( B \bigcup \left( \bigcup_{n=1}^{\infty} B_n \right) \right) \]

Since $\mu$ is countably weakly null-additive, we have
\[ \mu(D) = 0 \text{ for all } x \notin D \]
\[ \Rightarrow f(x) = \lim_{n \to \infty} f_n(x) \geq g \text{ for all } x \in D \]

Therefore
\[ f \geq 0 \text{ a.e.} \]

(2) Since $f_n \leq g$ a.e
\[ \Rightarrow f_n \geq 0 \text{ a.e.} \]

Since $f_n \xrightarrow{a.e} f$
Let $\mu$ is countably weakly null-additive, we have $\mu(D) = 0$ for all $x \notin D$, and
\[
f_n(x) \rightarrow f(x), \quad f_n(x) \rightarrow g(x) \quad \text{uniformly for any } x \notin D
\]
Since $\mu(D) = 0$ for all $x \notin D$
\[
\Rightarrow f(x) = g(x) \quad \text{for any } x \notin D
\]
\[\therefore f = g \text{ a.e.} \]

(2) Since $f_n \xrightarrow{a,u} f$, then there is a sequence of sets $\{A_n\}$ in $\mathcal{F}$ with $\lim_{n \to \infty} \mu(A_n) = 0$ such that $f_n \rightarrow f$ on $A/A_n$ for any fixed $n = 1, 2, \ldots$
I.e. $f_n(x) \rightarrow f(x)$ uniformly for any $x \in A/A_n$
Since $f = g \text{ a.e.}$, then there is a subset $B \subseteq A$ such that $\mu(B) = 0$ and $f(x) = g(x)$ for all $x \in A/B$
Let
\[
B_n = \emptyset, \quad \text{for all } n \geq 2
\]
\[
B_1 = B, B_2 = \emptyset, B_3 = \emptyset, \ldots
\]
\[
\bigcup_{n=1}^{\infty} B_n = B
\]
\[
\Rightarrow \mu(B) = \mu \left( \bigcup_{n=1}^{\infty} B_n \right)
\]
\[\therefore \mu \left( \bigcup_{n=1}^{\infty} B_n \right) = 0 \]

Let
\[
D_n = \left( \bigcup_{n=1}^{\infty} A_n \right) \cup \left( \bigcup_{n=1}^{\infty} B_n \right)
\]
Where $\{D_n\}$ be a sequence of sets in $\mathcal{F}$
\[
\lim_{n \to \infty} \mu(D_n) = \mu \left( \bigcup_{n=1}^{\infty} A_n \right) \cup \left( \bigcup_{n=1}^{\infty} B_n \right)
\]
Since $\mu$ is continuous from below at $A$, we have
\[
A = \bigcup_{n=1}^{\infty} A_n, \quad \mu(A) = \lim_{n \to \infty} \mu(A_n)
\]
\[
\mu(A) = 0
\]
\[\Rightarrow \mu(A) = \mu \left( \bigcup_{n=1}^{\infty} A_n \right)
\]
\[\Rightarrow \mu \left( \bigcup_{n=1}^{\infty} A_n \right) = 0
\]
\[\Rightarrow \lim_{n \to \infty} \mu(D_n) = \mu(A \cup B)
\]
Since $\mu$ is countably weakly null-additive, we have $\lim_{n \to \infty} \mu(D_n) = 0$ for any $x \notin D_n$
\[
f_n(x) \rightarrow f(x) = g(x) \quad \text{Uniformly for any } x \notin D_n
\]
Therefore
\[
f_n \xrightarrow{a,u} g.
\]

(3) Since $f_n \xrightarrow{a,u} f$, then there is a sequence of sets $\{A_n\}$ in $\mathcal{F}$ with $\lim_{n \to \infty} \mu(A_n) = 0$ such that $f_n \rightarrow f$ on $A/A_n$ for any fixed $n = 1, 2, \ldots$
I.e. $f_n(x) \rightarrow f(x)$ uniformly for any $x \in A/A_n$
Since $f = g \text{ a.e.}$ for each $n$, then there is a sequence $\{B_n\}$ in $\mathcal{F}$ such that
\[
\mu(B_n) = 0 \quad \text{and} \quad f_n(x) = g_n(x) \quad \text{for all } x \in A/B_n
\]
Let
\[
D_n = \left( \bigcup_{n=1}^{\infty} A_n \right) \cup \left( \bigcup_{n=1}^{\infty} B_n \right)
\]
Where $\{D_n\}$ be a sequence of sets in $\mathcal{F}$
\[
\lim_{n \to \infty} \mu(D_n) = \mu \left( \bigcup_{n=1}^{\infty} A_n \right) \cup \left( \bigcup_{n=1}^{\infty} B_n \right)
\]
Since $\mu$ is continuous from below at $A$, we have
\[
A = \bigcup_{n=1}^{\infty} A_n, \quad \mu(A) = \lim_{n \to \infty} \mu(A_n)
\]
\[\Rightarrow \mu(A) = \mu \left( \bigcup_{n=1}^{\infty} A_n \right)
\]
\[\Rightarrow \mu \left( \bigcup_{n=1}^{\infty} A_n \right) = 0
\]
Since $\mu$ is countably weakly null-additive, we have $\mu(B_n) = 0$, for all $n \geq 1$
\[\Rightarrow \mu \left( \bigcup_{n=1}^{\infty} B_n \right) = 0
\]
\[
\lim_{n \to \infty} \mu(D_n) = \mu \left( \bigcup_{n=1}^{\infty} A_n \right)
\]
\[\Rightarrow \mu \left( \bigcup_{n=1}^{\infty} A_n \right) = 0
\]
Also countably weakly null-additive, this mean
\[ \limsup_{n \to \infty} \mu(B_n) = 0 \text{ for all } n \geq 1 \]
\[ \Rightarrow \mu \left( \bigcup_{n=1}^{\infty} B_n \right) = 0 \]
Since is countably weakly null-additive
\[ \lim_{n \to \infty} \mu(D_n) = 0 \text{ for any } x \notin D_n \]
and \( g_n(x) = f_n(x) \to g(x) \) Uniformly for any \( x \in D_n \)
Therefore
\[ g_n(x) \to g(x) \text{ Uniformly for any } x \notin D_n \]
\[ \Rightarrow g_n \to g. \]

**Theorem (13):**
Let \( (\Omega, \mathcal{F}, \mu) \) be a fuzzy measure space such that \( \mu \) is countably weakly null-additive and continuous from below at \( A \), let \( f_n, g_n, f, g \in C(\Omega) \), \( n \in \mathbb{N} \) and let \( A \in \mathcal{F}, C \in \mathcal{R} \) if \( f_n \to f \) and \( g_n \to g \) then
\[ \begin{align*}
(1) & \quad c.f_n \to c.f. \\
(2) & \quad f_n + g_n \to f + g. \\
(3) & \quad |f_n| \to |f|.
\end{align*} \]

**Proof:**

(1) Since \( f_n \xrightarrow{a.s.} f \), then there is a sequence of sets \( \{A_n\} \) in \( \mathcal{F} \) with \( \lim_{n \to \infty} \mu(A_n) = 0 \) such that \( f_n \to f \) on \( A_n \) for any fixed \( n = 1, 2, \ldots \)
\[ \text{I.e.} f_n(x) \to f(x) \text{ uniformly for any } x \in A_n 
\Rightarrow c.f_n \to c.f. \]

(2) Since \( f_n \xrightarrow{a.s.} f \), then there is a sequence of sets \( \{A_n\} \) in \( \mathcal{F} \) with \( \lim_{n \to \infty} \mu(A_n) = 0 \) Such that \( f_n \to f \) on \( A_n \) for any fixed \( n = 1, 2, \ldots \)
I.e. \( f_n(x) \to f(x) \) uniformly for any \( x \in A_n \)
Since \( g_n \to g \), then there is a of sets \( \{B_n\} \) in \( \mathcal{F} \) with \( \lim_{n \to \infty} \mu(B_n) = 0 \) such that \( g_n \to g \) on \( B_n \) for any fixed \( n = 1, 2, \ldots \)
I.e. \( g_n(x) \to g(x) \) uniformly for any \( x \in B_n \)
Let
\[ D_n = \left( \bigcup_{n=1}^{\infty} A_n \right) \cup \left( \bigcup_{n=1}^{\infty} B_n \right) \]
Where \( \{D_n\} \) be a sequence of sets in \( \mathcal{F} \)
\[ \Rightarrow \lim_{n \to \infty} \mu(D_n) = \mu \left( \bigcup_{n=1}^{\infty} A_n \right) \cup \left( \bigcup_{n=1}^{\infty} B_n \right) \]
Since \( \mu \) is continuous from below at \( A \), we have
\[ \bigcup_{n=1}^{\infty} A_n = A, \mu(A) = \lim_{n \to \infty} \mu(A_n) \]
\[ \Rightarrow \lim_{n \to \infty} \mu(D_n) = 0 \]

**Theorem (14):**
Let \( (\Omega, \mathcal{F}, \mu) \) be a fuzzy measure space, let \( f, f_n \in C(\Omega), n \in \mathbb{N} \) and let \( A \in \mathcal{F} \) such that \( \mu \) is countably weakly null-additive and continuous from below at \( A \), if \( f_n \xrightarrow{a.s.} f \) then
\[ \begin{align*}
(1) & \quad \text{If } f_n \xrightarrow{\mu} g \text{ then } f = g \text{ a.e.} \\
(2) & \quad \text{If } f = g \text{ a.e. then } f_n \xrightarrow{\mu} g. \\
(3) & \quad \text{If } g_n = g \text{ a.e. for all then } g_n \to f.
\end{align*} \]

**Proof:**

(1) Given any \( \varepsilon > 0 \), define
\[ B = \{ x \in \Omega : |f(x) - g(x)| \geq \varepsilon \} \cap A \]
\[ B_n = \{ x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon \} \cap A \]
\[ C_n = \{ x \in \Omega : |f_n(x) - g(x)| \leq \varepsilon \} \cap A \]
Since
\[ |f(x) - g(x)| \leq |f_n(x) - f(x)| + |f_n(x) - g(x)| \]
This implies that
\[ B \subseteq B_n \cup C_n \Rightarrow \mu(B) \leq \mu(B_n \cup C_n) \]
Since \( f_n \to f, f_n \to g \)
\[ \Rightarrow \mu(B_n) \to 0, \mu(C_n) \to 0 \text{ as } n \to \infty \]
Since \( \mu \) is countably weakly null-additive, we have
\[ \mu(B_n \cup C_n) \to 0 \text{ as } n \to \infty \]
Therefore \( \mu(B) \to 0 \text{ as } n \to \infty \)
\[ N(f-g) = \{ x \in \Omega : (f-g)(x) \neq 0 \} \]
\[ = \bigcup_{n=1}^{\infty} \{ x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon \} \cap A \]
\[ \Rightarrow \mu(N(f-g)) = 0 \Rightarrow f = g \text{ a.e.} \]

(2) Since \( f = g \text{ a.e.} \) there exists \( B \in \mathcal{F} \) with \( \mu(B) = 0 \) and \( f(x) \neq g(x) \) for all \( x \in B \) for any \( \varepsilon > 0 \), we have
\[ \{ x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon \} \cap A \subseteq B \cup \{ x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon \} \cap A \]
\[ \Rightarrow \mu(\{ x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon \} \cap A) \leq \mu(B) \cup \{ x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon \} \cap A \]
Since \( f_n \xrightarrow{\mu} f \) then
\[ \mu(\{ x \in \Omega : |f_n(x) - f(x)| \geq \varepsilon \} \cap A) = 0, \mu(B) = 0 \]
Since \( \mu \) countably weakly null-additive, we have
\[ \Rightarrow \mu(\{ x \in \Omega : |f_n(x) - g(x)| \geq \varepsilon \} \cap A) = 0 \text{ as } n \to \infty \]
\[ \Rightarrow f_n \xrightarrow{\mu} g. \]

(3) Since \( f_n = g \text{ a.e. for all } n \), then there exists \( A_n \in \mathcal{F} \) with \( \mu(A_n) = 0 \) and \( f_n(x) \neq g_n(x) \) for all \( x \in A_n \)
Since \( \mu \) is discontinuous from below at \( A \), we have
\[
A = \bigcup_{n=1}^{\infty} A_n, \mu(A) = \lim_{n \to \infty} \mu(A_n)
\]
\[
\Rightarrow \mu(A) = \mu \left( \bigcup_{n=1}^{\infty} A_n \right)
\]
Since \( \mu \) is countably weakly null-additive
\[
\Rightarrow \mu(A) = 0, \text{for any } \varepsilon > 0,
\]
we have
\[
C = \left\{ (x \in \Omega: |g_n(x) - f(x)| \geq \varepsilon) \cap B \right\}
\]
\[
C_n = \left\{ (x \in \Omega: f_n(x) - f(x) \geq \frac{\varepsilon}{2}) \cap B \right\}
\]
\[
D_n = \left\{ (x \in \Omega: |g_n(x) - f(x)| \geq \frac{\varepsilon}{2}) \cap B \right\}
\]
Since
\[
|g_n(x) - f(x)| \leq f_n(x) - f(x) + |g_n(x) - f(x)|
\]
\[
\Rightarrow C \subseteq C_n \cup D_n \Rightarrow \mu(C) \leq \mu(C_n \cup D_n)
\]
Since \( f_n \to f, g_n \to g \)
\[
\Rightarrow \mu(C_n) \to 0, \mu(D_n) \to 0 \text{ as } n \to \infty
\]
Since \( \mu \) is countably weakly null-additive, we have
\[
\mu(C_n \cup D_n) \to 0 \text{ as } n \to \infty
\]
Therefore
\[
\Rightarrow g_n \to f.
\]
**Theorem (15):**
Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space such that \( \mu \) is weakly null-additive, let \( f, g, f_n, g_n \in C(\Omega), n \in \mathbb{N} \) and \( A \in \mathcal{F}, \) if \( f_n \to f, g_n \to g \) \( \in \mathbb{R} \), then
\[
(1) \ c \cdot f_n \to c \cdot f
\]
\[
(2) \ \|f_n - f\| \to 0
\]
**Proof:**
(1) This is clear if \( c = 0, \) if \( c \neq 0, \) let \( \varepsilon > 0.\)
Since \( f_n \to f \) and
\[
\left\{ x \in \Omega: |c \cdot f_n(x) - c \cdot f(x)| \geq \varepsilon \right\} \cap A
\]
\[
= \left\{ x \in \Omega: f_n(x) - f(x) \geq \frac{\varepsilon}{|c|} \right\} \cap A
\]
This implies that
\[
\mu \left( \left\{ x \in \Omega: f_n(x) - f(x) \geq \frac{\varepsilon}{|c|} \right\} \cap A \right) = 0 \text{ as } n \to \infty
\]
So that
\[


\]
(2) Since
\[
\|f_n(x) - f(x)\| \leq |f_n(x) - f(x)|
\]
This implies that
\[
\left\{ x \in \Omega: |f_n(x) - f(x)| \geq \varepsilon \right\} \cap A
\]
\[
\subseteq \left\{ x \in \Omega: f_n(x) - f(x) \geq \varepsilon \right\} \cap A
\]
Since \( f_n \to f \) so
\[
\mu \left( \left\{ x \in \Omega: |f_n(x) - f(x)| \geq \varepsilon \right\} \cap A \right) = 0 \text{ as } n \to \infty
\]
Therefore
\[
\|f_n \| \to 0.\]
**Theorem (16):**
Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space such that \( \mu \) is weakly null-additive, let \( f, g, f_n, g_n \in C(\Omega), n \in \mathbb{N} \) and \( A \in \mathcal{F}, \) suppose that \( f_n + g_n \to 0 \) whenever \( f_n \to 0 \) and \( g_n \to 0, \) then \( \mu \) is autocontinuous from below
**Proof:**
Let \( \{A_n\} \) be a sequence with \( \lim_{n \to \infty} \mu(A_n) = 0, \) given any \( \varepsilon > 0. \)

Suppose \( \mu \) is not autocontinuous from below
Take \( \lim_{n \to \infty} \mu(A \cup A_n) > 0 \)
There is no loss of generality \( A, A_n \in \mathcal{F} \) and \( A \cap A_n = \emptyset \)
\[
f_n(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}
\]
and
\[
g_n(x) = \begin{cases} 0 & \text{if } x \notin A_n \\ 1 & \text{if } x \in A_n \end{cases}
\]
Then \( f_n \to 0 \) and \( g_n \to 0, \) thus
\[
f_n + g_n \to 0, \text{ if } x \notin A \cup A_n
\]
So \( f_n + g_n \to 0 \)
\[
\lim_{n \to \infty} \mu(A \cup A_n) = \lim_{n \to \infty} \mu((x \in \Omega: |f_n(x) + g_n(x)| \geq \varepsilon) \cap A) = 0
\]
\[
\Rightarrow \lim_{n \to \infty} \mu(A \cup A_n) = 0
\]
Which is contradiction with assumption that \( \lim_{n \to \infty} \mu(A \cup A_n) > 0 \)
Consequently \( \mu \) is autocontinuous from below

**Theorem (17):**
Let \((\Omega, \mathcal{F}, \mu)\) be a fuzzy measure space such that \( \mu \) is weakly null-additive, let \( f, g, f_n, g_n \in C(\Omega), n \in \mathbb{N} \) and \( A \in \mathcal{F}, \) suppose that \( f_n \to 0 \) and \( g_n \to 0, \) whenever \( f_n \to 0 \) and \( g_n \to 0, \) then \( \mu \) is null continuous.

**Proof:**
Let \( A_1 = \left\{ x \in \Omega: |f_n(x)| \geq \frac{\varepsilon}{2} \right\} \cap A \)
\[
A_2 = \left\{ x \in \Omega: |g_n(x)| \geq \frac{\varepsilon}{2} \right\} \cap A \]
\[
A_3 = \left\{ x \in \Omega: |f_n(x) + g_n(x)| \geq \varepsilon \right\} \cap A \]
Since \( f_n \to 0, g_n \to 0 \) and \( f_n + g_n \to 0, \) we have
\[
\lim_{n \to \infty} \mu \left( \left\{ x \in \Omega: |f_n(x)| \geq \frac{\varepsilon}{2} \right\} \cap A \right) = 0
\]
\[
\lim_{n \to \infty} \mu \left( \left\{ x \in \Omega: |g_n(x)| \geq \frac{\varepsilon}{2} \right\} \cap A \right) = 0
\]
\[
\lim_{n \to \infty} \mu \left( \left\{ x \in \Omega: |f_n(x) + g_n(x)| \geq \varepsilon \right\} \cap A \right) = 0
\]
\[
\Rightarrow \mu(A_1) = 0, \mu(A_2) = 0, \mu(A_3) = 0
\]
\[
\therefore \mu \text{ is countably weakly null-additive, for all } n \geq 1
\]
\[
\Rightarrow \mu \left( \bigcup_{n=1}^{\infty} A_n \right) = 0
\]
\[
\therefore \mu \text{ is null-continuous.}
\]

**References**


[5] De Li Zhang, Caiwei Guo "on the convergence of..."


