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BMAP Two Phase Retrial Queue with Feedback

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Abstract: ***This paper is concerned with the analysis of a single server batch arrival retrial queue with general retrial times. The server provides two phases of service – essential and optional. A customer after receiving essential service may leave the system or rejoin the orbit and begin requesting service again. The model for the system is derived and the joint distribution of the server state and the orbit length in steady state is obtained. Numerical examples are presented to illustrate the effect of the parameters on several performance measures.***

Keywords: Retrial queues, batch arrival, probability generating function, steady state, feedback

1. Introduction

Retrial queueing system are characterized by the feature that the arrivals who find a free server enters into service immediately; otherwise the customer enters into an orbit. An orbiting customer competes for service by sending out signals at random times until a free server is captured. Retrial queues are widely used in mathematical models of several computer and telecommunication systems.

There have been several contributions considering queueing systems of M/G/1 type in which the server may provide a second phase of service. Medhi [9] has studied an M/G/1 queue where the server provides the first essential service to all the arriving customers, whereas only some of them receive a second option service. Choudhury [4] has obtained the waiting time distribution for the queueing model discussed by Medhi [10]. Krishna Kumar et al. [8, 9] have analyzed an M/G/1 retrial queue with two phase services, preemptive priority and single server feedback retrial queue respectively.

Feedback queues relate to those queues in which a customer served once, when his service becomes unsuccessful and is served again and again till his service becomes successful. Many authors Takacs [11], Folely et al. [6] and Workman et al. [12] have analyzed queueing model with feedback. These models are motivated mainly by application in data transmission, manufacturing processes where quality control inspections are performed and so on. In this paper an $M^X/G/1$ queue with additional optional service and customers feedback is analyzed for its performance measures.

2. Model Description

Consider a single server retrial queue where optional service is provided after the essential service. The server provides essential service to all the arriving customers. Customers arrive in groups according to a Poisson process with rate λ. The batch size Y is a random variable and $P(Y=k) = C_k$, k = 1,2,3,… with $\sum_k C_k = 1$.

Denote by $C(z) = \sum C_k z^k$ the generating function of the batch size distribution with first two moments τ_1 and τ_2. If the server is free then the essential service commences for one of the arriving customers and the others join the orbit. While at the essential service, the server may push out (with probability α) the customer under going service to the orbit and start serving an arriving customer or continue the ongoing service (with probability $1-\alpha$) so that all the arriving customers join the orbit. Successive inter retrial times of any customer are governed by an arbitrary probability distribution function A (.) with corresponding density function a (.) and Laplace Stieltjes transform A*(.).

The server provides the essential service to all arriving customers. Let B (.) and b (.) be respectively the cumulative distribution and the probability density function of the essential service time with Laplace Stieltjes transform B*(.). After the completion of phase 1 service, the customer may leave the system with probability δ or go for optional service with probability β or return back to the orbit with probability γ $(= 1-\beta-\delta)$. The optional service times of customers are independent random variables with common distribution function B_1 (.), probability density function b_1 (.), Laplace Stieltjes transform $B_1^*(.)$ and first two moments h_1 and h_2.

The stochastic behaviour of this retrial queueing system can be described by the Markov process $\{N(t), t \geq 0\} = \{(C(t), X(t), \xi_0(t), \xi_1(t), \xi_2(t), t \geq 0\}$ where C (t) denotes the server state 0, 1 or 2 according as the server being idle, busy with essential service, busy with optional service and X (t) corresponds to the number of customers in the orbit at time t. If C (t) = 0 and X (t) > 0, then $\xi_0(t)$ represents the elapsed retrial time. If C (t) = 1 and X (t) $\geq$ 0, then $\xi_1(t)$ corresponds to the elapsed time of the customer being provided phase 1 service. If C (t) = 2 and X (t) $\geq$ 0, then $\xi_2(t)$ represents the elapsed time of the customer being provided optional service. The functions $\eta(x)$, $\mu(x)$ and $\mu_1(x)$ are the conditional completion rates for repeated attempts, essential service and for optional service respectively. Then $\eta(x) = a(x) / [1 - A(x)]$;
$\mu(x) = b(x) / [1 - B(x)]$ and $\mu_1(x) = b_1(x) / [1-B_1(x)]$.

3. Theorem

Let X_n, $n \geq 1$ be the orbit length at the time of the n^{th} customers departure . Then $\{X_n, n \geq 1\}$ is ergodic if and only if $\tau_1 [1 - B^*(\alpha\lambda)] / (\alpha B^*(\alpha\lambda)) + \gamma + \beta\lambda\tau_1 h_1 + \tau_1 [1 - A^*(\lambda)] < 1$.

The theorem can be proved along similar lines as in Gomez-Corral [7].

As the arrival stream is a Poisson process with mean batch size τ_1, it can be shown from Burke's theorem [5] that the steady state probabilities of $\{C(t), X(t), t \geq 0\}$ exist and they are positive if and only if
$\tau_1 [1 - B^*(\alpha\lambda)] / (\alpha B^*(\alpha\lambda)) + \gamma + \beta\lambda\tau_1 h_1 + \tau_1 [1 - A^*(\lambda)] < 1$.

From the mean drift $\chi_j = \tau_1 [1 - B^*(\alpha\lambda)] / (\alpha B^*(\alpha\lambda)) + \gamma + \beta\lambda\tau_1 h_1 + \tau_1 [1 - A^*(\lambda)] - 1$, for $j \geq 1$, we have the reasonable conclusion that the first two terms represents the mean number of customers leaving for orbit due to the decision of the server to push out or continue the ongoing service or due to feedback of customers. The term $\tau_1\lambda\beta h_1$ represent the batch arrival during service time in the optional service leaving for the orbit. The last term $\tau_1 [1 - A^*(\lambda)] - 1$ refers to the contribution to the orbit size due to a batch arrival during the retrial time excluding the arbitrary customer of the arriving batch whose service commences immediately. Thus, the condition $\chi_j < 0$ ensures that the orbit size does not grow indefinitely in course of time.

4. Steady State Distribution

In this section the steady state distribution of the system are derived. For the process $\{N(t), t \geq 0\}$, define the probability

$$I_0(t) = P\{C(t) = 0, X(t) = 0\}$$

and the probability densities

$$I_n(x, t)\,dx = P\{C(t) = 0, X(t) = n, x \leq \xi_0(t) < x + dx\}, \quad t \geq 0, x \geq 0 \text{ and } n \geq 1$$

$$W_n(x, t)\,dx = P\{C(t) = 1, X(t) = n, x \leq \xi_1(t) < x + dx\}, \quad t \geq 0, x \geq 0 \text{ and } n \geq 0$$

$$S_n(x, t)\,dx = P\{C(t) = 2, X(t) = n, x \leq \xi_2(t) < x + dx\}, \quad t \geq 0, x \geq 0 \text{ and } n \geq 0$$

By supplementary variable technique, the system of equations that governs the model are given by

$$\frac{d I_0(t)}{dt} = -\lambda I_0(t) + \delta \int_0^\infty W_0(x, t)\,\mu(x)\,dx + \int_0^\infty S_0(x, t)\,\mu_1(x)\,dx \qquad (1)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) I_n(x, t) = -(\lambda + \eta(x))\, I_n(x, t), \quad n \geq 1 \qquad (2)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) W_n(x, t) = -(\lambda + \mu(x))\, W_n(x, t) + \lambda(1 - \alpha)(1 - \delta_{0n}) \sum_{k=1}^{n} C_k\, W_{n-k}(x, t), \quad n \geq 0 \qquad (3)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) S_n(x, t) = -(\lambda + \mu_1(x))\, S_n(x, t) + \lambda(1 - \delta_{0n}) \sum_{k=1}^{n} C_k\, S_{n-k}(x, t), \quad n \geq 1 \qquad (4)$$

with boundary conditions

$$I_n(0, t) = \delta \int_0^\infty W_n(x, t)\,\mu(x)\,dx + \int_0^\infty S_n(x, t)\,\mu_1(x)\,dx + \gamma \int_0^\infty W_{n-1}(x, t)\,\mu(x)\,dx, \quad n \geq 1 \qquad (5)$$

$$W_0(0, t) = \lambda C_1 I_0(t) + \int_0^\infty I_1(x, t)\,\eta(x)\,dx \qquad (6)$$

$$W_n(0, t) = \lambda \int_0^\infty \sum_{k=1}^{n} C_k\, I_{n-k+1}(x,t)\,dx + \int_0^\infty I_{n+1}(x, t)\,\eta(x)\,dx + \lambda C_{n+1} I_0(t) + \lambda\alpha \int_0^\infty \sum_{k=1}^{n} C_k\, W_{n-k}(x, t)\,dx, \quad n \geq 1 \qquad (7)$$

$$S_n(0, t) = \beta \int_0^\infty W_n(x, t)\,\mu(x)\,dx, \quad n \geq 0 \qquad (8)$$

The steady state equations corresponding to the equations (1) through (8) are given by

$$\lambda I_0 = \delta \int_0^\infty W_0(x)\,\mu(x)\,dx + \int_0^\infty S_0(x)\,\mu_1(x)\,dx \qquad (9)$$

$$\frac{d I_n(x)}{dx} = -(\lambda + \eta(x))\, I_n(x), \quad n \geq 1 \qquad (10)$$

$$\frac{d W_n(x)}{dx} = -(\lambda + \mu(x))\, W_n(x) + \lambda(1 - \alpha)(1 - \delta_{0n}) \sum_{k=1}^{n} C_k\, W_{n-k}(x), \quad n \geq 0 \qquad (11)$$

$$\frac{d S_n(x)}{dx} = -(\lambda + \mu_1(x))\, S_n(x) + \lambda(1 - \delta_{0n}) \sum_{k=1}^{n} C_k\, S_{n-k}(x), \quad n \geq 0 \qquad (12)$$

With boundary conditions

$$I_n(0) = \delta \int_0^\infty W_n(x)\,\mu(x)\,dx + \gamma \int_0^\infty W_{n-1}\,\mu(x)\,dx + \int_0^\infty S_n(x)\,\mu_1(x)\,dx, \quad n \geq 1 \qquad (13)$$

$$W_0(0) = \lambda C_1 I_0 + \int_0^\infty I_1(x)\,\eta(x)\,dx \qquad (14)$$

$$W_n(0) = \lambda \int_0^\infty \sum_{k=1}^{n} C_k\, I_{n-k+1}(x)\,dx + \int_0^\infty I_{n+1}(x)\,\eta(x)\,dx + \lambda C_{n+1} I_0$$

$$+\lambda\alpha\int_0^\infty \sum_{k=1}^{n} C_k\, W_{n-k}(x)\,dx, \quad n\ge 1 \qquad (15)$$

$$S_n(0) = \beta\int_0^\infty W_n(x)\,\mu(x)\,dx, \quad n\ge 0 \qquad (16)$$

Define the probability generating function
$P(., z) = \sum_n p_n(.)\, z^n$ for any probability p_n

The steady state distributions of $\{N(t), t \ge 0\}$ are given by

$$I(x, z) = I(0, z)\, e^{-\lambda x}\, [1 - A(x)] \qquad (17)$$

$$W(x, z) = W(0, z)\, e^{-[\lambda(1 - C(z) + \alpha C(z))]\, x}\, [1 - B(x)] \qquad (18)$$

$$S(x, z) = S(0, z)\, e^{-\lambda(1 - C(z))x}\, [1 - B_1(x)] \qquad (19)$$

$$I(0, z) = [\delta + \gamma z + \beta B_1^*(\lambda(1 - C(z)))]\, B^*(\lambda(1 - C(z) + \alpha C(z)))\, W(0, z) - \lambda I_0 \qquad (20)$$

$$S(0, z) = \beta B^*(\lambda(1 - C(z) + \alpha C(z)))\, W(0, z) \qquad (21)$$

$$W(0, z) = \lambda I_0 A^*(\lambda)\, [1 - C(z)]\, [1 - C(z) + \alpha C(z)] / D(z) \qquad (22)$$

Where

$$D(z) = B^*(\lambda(1-C(z) + \alpha C(z)))[\delta + \gamma z + \beta B_1^*(\lambda(1-C(z)))]\,[1-C(z) + \alpha C(z)]\,[C(z) + A^*(\lambda)(1-C(z))] - \alpha C(z)\, B^*(\lambda(1-C(z) + \alpha C(z))) - z\,[1-C(z)] \qquad (23)$$

Define the partial generating function $\psi(z) = \int_0^\infty \psi(x, z)\,dx$ for any generating function $\psi(x, z)$. Then

$$I(z) = \frac{I_0}{D(z)}(1 - A^*(\lambda))\{z(1 - C(z) + B^*(\lambda(1 - C(z) + \alpha C(z)))\, C(z)\,[\alpha z - (1 - C(z) + \alpha C(z))(\delta + \gamma z + \beta B_1^*(\lambda(1 - C(z))))]\} \qquad (24)$$

$$W(z) = \frac{I_0}{D(z)} A^*(\lambda)(1-C(z))[1-B^*(\lambda(1-C(z)+\alpha C(z)))] \qquad (25)$$

$$S(z) = \frac{I_0}{D(z)} A^*(\lambda)\,\beta\,[1 - C(z) + \alpha C(z)]\, B^*(\lambda(1 - C(z) + \alpha C(z)))\,[1 - B_1^*(\lambda(1 - C(z)))] \qquad (26)$$

Using the normalizing condition, I_0 is obtained as

$$I_0 = \{B^*(\alpha\lambda)\,[(1 - \alpha)\tau_1 + \alpha(\tau_1 A^*(\lambda) + 1 - \gamma - \beta\lambda\tau_1 h_1)] - \tau_1\} / \{\alpha(1 - \gamma)\, A^*(\lambda)\, B^*(\alpha\lambda)\} \qquad (27)$$

The probability generating function K(z) for the number of customers in the system is

$$K(z) = I_0 + I(z) + z\,W(z) + z\,S(z)$$
$$= I_0 A^*(\lambda)\, B^*(\lambda(1-C(z) + \alpha C(z))(1-z)\,[1-C(z) + \alpha C(z)]\,[\delta + \beta B_1^*(\lambda(1-C(z)))] / D(z) \quad (28)$$

The probability generating function H(z) for the number of customers in the orbit is

$$H(z) = I_0 + I(z) + W(z) + S(z)$$
$$= I_0 A^*(\lambda)(1 - z)\{B^*(\lambda(1 - C(z) + \alpha C(z)))\,[\alpha C(z)(1 - \gamma) - (1 - C(z))\gamma] + (1-C(z))\} / D(z)$$

5. Performance Measures

In this section some performance measures for the system under steady state are derived.

The probability that the server is idle during the retrial time is given by

$$I(1) = [1 - A^*(\lambda)]\,[\tau_1 [1-B^*(\alpha\lambda)(1 - \alpha - \alpha\beta\lambda h_1)] - \alpha(1-\gamma)B^*(\alpha\lambda)] / [\alpha(1-\gamma)A^*(\lambda)\, B^*(\alpha\lambda)]$$

The probability that the server is busy for providing essential service is given by

$$W(1) = \tau_1 [1 - B^*(\alpha\lambda)] / [\alpha(1 - \gamma)\, B^*(\alpha\lambda)]$$

The probability that the server is busy for providing optional service is given by

$$S(1) = \tau_1 \beta\lambda h_1 / (1 - \gamma)$$

Take

$$D_1 = \tau_1 [B^*(\alpha\lambda)\,[1 - \alpha(1- A^*(\lambda) + \beta\lambda h_1)] - 1] + \alpha(1 - \gamma)\, B^*(\alpha\lambda)$$

$$D_2 = \lambda\tau_1 (1 - \alpha)\, B^{*\prime}(\alpha\lambda)\,[\alpha(\gamma + \beta\lambda\tau_1 h_1) - \tau_1(1 - \alpha) - \alpha - \alpha\tau_1 A^*(\lambda)] + \tau_1 + B^*(\alpha\lambda)\,[\alpha\tau_1(1-A^*(\lambda))\,[\gamma + \beta\lambda\tau_1 h_1] - \tau_1^2 (1-\alpha)(1-A^*(\lambda)) - \alpha\tau_1 - \tau_1(1-\alpha)(\gamma + \beta\lambda\tau_1 h_1)] + (B^*(\alpha\lambda)\,[\alpha\tau_2(1 - A^*(\lambda)) - \tau_2 + \alpha\beta\lambda\tau_2 h_1 + \alpha\beta\lambda^2\tau_1^2 h_2] + \tau_2) / 2$$

$$N_1 = \tau_1 [\alpha\lambda(1 - \alpha)\, B^{*\prime}(\alpha\lambda)(1 - \gamma) + B^*(\alpha\lambda)(\alpha\beta\lambda h_1 - (1 - \gamma)(1 - \alpha))]$$

$$N_2 = \tau_1 [\alpha\lambda(1 - \alpha)(1 - \gamma)\, B^{*\prime}(\alpha\lambda) + B^*(\alpha\lambda)(\gamma + \alpha(1 - \gamma)) - 1]$$

The mean number of customers in the system is

$$L_s = K'(1) = N_1 / [\alpha(1 - \gamma)\, B^*(\alpha\lambda)] + D_2 / D_1$$

The mean number of customer in the orbit is

$$L_q = H'(1) = N_2 / [\alpha(1 - \gamma)\, B^*(\alpha\lambda)] + D_2 / D_1.$$

6. Numerical Results

Table 1 presents the values of expected system size L_s, expected queue size L_q, the probability that the server is idle during retrial time I (1), the probability that the server is busy for providing essential service W (1) and the probability that the server is busy for providing optional service S (1)for fixed values of C_1=0.5, C_2=0.5, α=0.2, β=0.1, λ=1 and various values of γ, η, μ_1 and μ_2.
The following results are observed from the table 1.

- System size L_s directly proportional to the feedback probability, service rates of essential and optional
- The same trend is observed with respect to the orbit size L_q
- The probability to have the server idle during retrial time increases with increase of feedback and decreases with increase in η, μ_1,μ_2.
- W (1), the probability that the server is busy for providing essential service increases for increase in γ, decreases for increase in μ_1 and constant for the variation in μ_2 and η.
- The influence of the two parameters η and μ_1 are not felt in S (1).

Table 1: Parameters influence on performance measures

	L_s	L_q	I (1)	W (1)	S (1)
γ					
0.2	0.4910	0.3973	0.1938	0.0750	0.0188
0.4	1.0226	0.8976	0.3250	0.1000	0.0214
0.6	4.3620	4.1745	0.5875	0.1500	0.0375

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η					
4	14.9077	14.7202	0.7344	0.1500	0.0375
8	1.5868	1.3993	0.3672	0.1500	0.0375
12	0.9758	0.7883	0.2448	0.1500	0.0375
μ_1					
12	41.1370	40.7870	0.6200	0.3125	0.0375
16	8.9910	8.7192	0.6044	0.2344	0.0375
20	5.7764	5.5514	0.5950	0.1875	0.0375
24	4.5571	4.3633	0.5888	0.1563	0.0375
28	3.9153	3.7438	0.5863	0.1442	0.0375
μ_2					
4	6.7466	6.5028	0.5988	0.1500	0.0938
8	4.6474	4.4506	0.5894	0.1500	0.0469
12	4.1885	4.0073	0.5863	0.1500	0.0313
16	3.9885	3.8151	0.5847	0.1500	0.0234
20	3.8766	3.7079	0.5838	0.1500	0.0188

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