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Chain Conditions on Fuzzy G-Modules

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Abstract: ***The concept of fuzzy G- Modules and its properties including complete reducibility,semi simplicity etc. are already defined. In this paper, the concept of minimal and maximal sub modules is defined on fuzzy G–modules. The chain conditions for fuzzy G-modules are also defined and some properties of both ascending and descending chains of sub modules are established.***

Keywords: Fuzzy-G-Modules, Proper fuzzy G sub modules, minimal and maximal sub modules of a fuzzy G module, chain conditions for fuzzy G modules

1. Introduction

The introduction of fuzzy sets by Zadeh led way to the fuzzification of Algebraic structures. Fuzzy groups and groupoids are defined by Rosenfield[9]. The concept of fuzzy G modules was introduced by the Sherry Fernandez [10]. Semi simplicity of fuzzy G modules and its relations with complete reducibility and injectivity were studied by the author [8],.

In this paper the concept of minimal and maximal sub modules is defined on fuzzy G–modules. The descending and ascending chain conditions for fuzzy G-modules are defined and some properties of these chains of fuzzy G sub modules are established.

2. Preliminaries

A vector space M over a field K is said to be a **G-Module** if for every g ∈ G and m ∈ M there exists a product 'gm' called action of G on M satisfying

(i) $1_G m = m, \forall m \in M$

(ii) $(gh)m = g(hm), \forall g, h \in G, m \in M$

(iii)$g(k_1m_1 + k_2m_2) = k_1(gm_1) + k_2(gm_2), \forall g \in G,\ m1,m2 \in M,\ k1,k2 \in K$

As an example, for $G = \{1, -1, i, -i\}$, $M = C^n$ over C is a G-Module under usual addition and multiplication.

A subspace of M, which itself is a G-Module with the same action is called **G sub module**. A non-zero G Module M is **irreducible** if the only G sub modules of M are M and $\{0\}$.otherwise it is **reducible.** A non-zero G module M is **completely reducible** if for every G sub module N of M there exists a G sub module N^* of M such that M=$N \oplus N^*$. A G Module M is **semi simple** if there exists a family of irreducible G sub modules M_i such that $M = {}_{i=1}^{n}\oplus M_i$

A fuzzy G module over a G Module M is a fuzzy set μ on M such that

(i) $\mu(ax+by) \geq \min(\mu(x), \mu(y)), \forall a, b \in K \text{ and } x, y \in M$

(ii) $\mu(gm) \geq \mu(m), \forall\, m \in M \text{ and } g \in G.$

For example, For $G = \{1, -1\}$ and $M = Q\sqrt{2}$ over Q , The fuzzy set μ on M defined as

$\mu(a+b\sqrt{2}) = 1$ if a = 0,b = 0

= .8 if a $\neq$ 0,b = 0

= .2 if b$\neq$ 0 is a fuzzy G module over M.

The standard fuzzy intersection of finite no of fuzzy G Modules is again a fuzzy G module. The collection of all elements in the universal set with membership value greater than a given α, $\alpha \in [0,1]$ is called an $\boldsymbol{\alpha}$ **cut** of the fuzzy G Module μ, denoted by $\mu^{\alpha+}$. The 0 cut, μ^{0+} consisting of all elements with a non zero membership is called the **support of** $\boldsymbol{\mu}$, denoted by $Supp\ \mu$.

3. Minimal and Maximal Fuzzy G sub modules

3.1 Definition

A fuzzy G sub module ϑ of a fuzzy G module ▯on a G module M is called a minimal fuzzy G sub module if $\vartheta \neq \chi_{\{0\}}$and $Supp\ \vartheta \subseteq Supp\ \theta$ for all fuzzy G sub modules θ of ▯.

3.2 Definition

A fuzzy G sub module ϑ of a fuzzy G module μ on a G module M is called a maximal fuzzy G sub module if $\vartheta \neq$ μ and $Supp\ \vartheta \supseteq Supp\ \theta$ for all fuzzy G sub modules θ of μ.

4. Descending and Ascending Chain Conditions on Fuzzy G sub modules

4.1 Definition

A fuzzy G module ▯ of a G Module M is said to satisfy the **descending chain condition (d.c.c)** if for any descending chain $\{\mu_i\}$ of fuzzy G sub modules of ▯ , the corresponding chain of supports, $\{Supp\mu_i\}$ satisfies the descending chain condition for G modules. There exists, an integer r satisfying $Supp\ \mu_r = Supp\ \mu_{r+1} = Supp\ \mu_{r+2} = \dots$

A fuzzy G module μ of a G Module M is said to satisfy the **ascending chain condition (a.c.c)** if for any ascending chain $\{\mu_i\}$ of fuzzy G sub modules of μ , the corresponding chain of supports $\{Supp\mu_i\}$ satisfies the ascending chain condition for G modules. Hence $Supp\mu_r = Supp\mu_{r+1} = Supp\mu_{r+2} = \dots$ for some integer r

4.2 Proposition

The following statements are equivalent for a fuzzy G module □ of a G Module M

1) The descending chain condition holds for □
2) Any non empty family of fuzzy G sub modules of□ has a minimal sub module

Proof: **1)** ⇒ **2)** Assume that descending chain condition holds for □. Consider a chain $\{\mu_i\}$ of fuzzy G sub modules of □. Take any fuzzy G sub module μ_i form the chain. If μ_i is the minimal then the proof is complete. Otherwise, there is fuzzy G sub module μ_j with $Supp\mu_j \subseteq Supp\mu_i$. If μ_j is the minimal one, the proof is complete. If not, we continue the process and if we cannot find a minimal element in finite steps, we get chain of fuzzy G sub modules whose corresponding chain of supports is non stationary. This contradicts our assumption and proves the existence of a minimal fuzzy G sub module for μ.

1) ⇒ **2)** Assume that, every non empty family of fuzzy G sub modules of µ has a minimal element. Hence there is a fuzzy G sub module μ_i, such that for any fuzzy G sub module $\mu_j$in the family , $Supp\mu_i \subseteq Supp\mu_j$. Consider any chain of fuzzy G sub modules of µ. Then it must have a minimal element μ_r satisfying $Supp\mu_r \subseteq Supp\mu_j$ for all $j \geq r$. But as the chain is a descending chain we can conclude that $Supp\,\mu_r = Supp\,\mu_j$ for all$j \geq r$. Hence μ satisfies the descending chain condition.

4.3 Proposition

If a fuzzy G module µ on a G module M satisfies descending chain condition then all its fuzzy G sub modules and quotient fuzzy G sub modules satisfy the d.c.c.

Proof:
Assume ϑ is a fuzzy G sub module of μ. Then any descending chain $\{\vartheta_i\}$ of fuzzy G sub modules of ϑ is a descending chain of fuzzy G sub modules of μ. Hence $\{supp\,\vartheta_i\}$ satisfies the d.c.c and thereby proves that the fuzzy G sub module ϑ of μ satisfies d.c.c.

Let $\mu_{\left(\frac{M}{N}\right)}$ is the quotient fuzzy G sub module of μ on $\frac{M}{N}$. Then any descending chain $\{\mu_{i\left(\frac{M}{N}\right)}\}$ of fuzzy G sub modules of $\mu_{\left(\frac{M}{N}\right)}$ corresponds to a descending chain of fuzzy G sub modules $\{\mu_i\}$ of μ with $supp\mu_{i\left(\frac{M}{N}\right)} = supp\,\mu_i$. As μ satisfies d.c.c, $\{supp\,\mu_i\}$ satisfies d.c.c of G modules. This proves that $\mu_{\left(\frac{M}{N}\right)}$ satisfies d.c.c.

4.4 Proposition

Let µ is a fuzzy G module on a G module M and ϑ is a fuzzy G sub module of µ defined on a G sub module N of M. If ϑ and $\mu_{\left(\frac{M}{N}\right)}$ satisfies d.c.c then µ also satisfies d.c.c.

Proof: Consider a descending chain $\{\mu_i\}$ of fuzzy G sub modules ofµ. Then $\{\mu_i \cap \vartheta\}$ is descending chain of fuzzy G sub modules of ϑ. As ϑ satisfies d.c.c, there is an integer r such that $supp(\mu_r \cap \vartheta) = supp(\mu_{r+1} \cap \vartheta) = supp\,(\mu_{r+2} \cap \vartheta) = \cdots$

Also $\{\mu_i + \vartheta\}_{\left\{\frac{supp\,(\mu_i+\vartheta)}{N}\right\}}$ is descending chain of fuzzy G sub modules of $\mu_{\left(\frac{M}{N}\right)}$. As $\mu_{\left(\frac{M}{N}\right)}$satisfies d.c.c there is an integer s such that $\frac{supp\,(\mu_s+\vartheta)}{N} = \frac{supp\,(\mu_{s+1}+\vartheta)}{N} = \frac{supp\,(\mu_{s+2}+\vartheta)}{N} = \cdots$

Hence we have

1. $\mu_n \geq \mu_{n+1}$for all n,
2. $supp(\mu_n \cap \vartheta) = supp(\mu_{n+1} \cap \vartheta)$ for all $n \geq r$,
3. $\frac{supp\,(\mu_n+\vartheta)}{N} = \frac{supp\,(\mu_{n+1}+\vartheta)}{N}$ for all $n \geq s$,
4. $supp(\mu_n + \vartheta) \cong \frac{supp\,\mu_n}{Supp\,\{\mu_n \cap \vartheta\}}$

Hence $\frac{supp\,\mu_n}{Supp\,\{\mu_n \cap \vartheta\}}\_\frac{supp\,(\mu_n+\vartheta)}{N} = \frac{supp\,(\mu_{n+1}+\vartheta)}{N} = \frac{supp\,\mu_{n+1}}{Supp\,\{\mu_{n+1} \cap \vartheta\}}$
and by using 2 we prove that $supp(\mu_n) = supp(\mu_{n+1})$ for all $n > \max(r, s)$. Hence satisfies d.c.c.

4.5 Proposition

The direct sum of finitely many fuzzy G sub modules on a G module M satisfying d.c.c satisfies d.c.c

Proof: Let $\mu_1, \mu_2, \ldots, \mu_n$ be a finite family of fuzzy G sub modules on a G module M satisfying d.c.c. We use Mathematical Induction to show that the direct sum $\mu = \oplus i{=}1n$µi of these n fuzzy G sub modules also satisfy d.c.c.
Let N = $\oplus_{i=1}^{n} supp\,\mu_i$. For $n = 1$, the proof is trivial. For $n \geq 2$ we assume that $\oplus_{i=1}^{n-1}\mu_i$ satisfies d.c.c. Then, $N' = \oplus_{i=1}^{n-1} supp\,\mu_i$ satisfies d.c.c.
We have, $\frac{N}{supp\,\mu_n} = \frac{N' + supp\,\mu_n}{supp\,\mu_n} = \frac{N'}{N' \cap supp\,\mu_n}$.
As N' satisfy d.c.c, $\frac{N'}{N' \cap supp\,\mu_n}$ satisfies d.c.c. Hence $\frac{N}{supp\,\mu_n}$ satisfies d.c.c. Since $supp\mu_n$ and $\frac{N}{supp\,\mu_n}$ satisfies d.c.c Nalso satisfies d.c.c. Since $supp\oplus_{i=1}^{n}\mu_i = \oplus_{i=1}^{n} supp\mu_i = N$, the proof is complete by mathematical induction.

4.6 Proposition

The following statements are equivalent for a fuzzy G module□ of a G Module M

1) The ascending chain condition holds for □
2) Any non empty family of fuzzy G sub modules of□ has a maximal sub module

Proof: **1)** ⇒ **2)** Assume that ascending chain condition holds for µ. Consider a family $\{\mu_i\}$ of fuzzy G sub modules of µ. Take any fuzzy G sub module μ_i form the family. If μ_i is the maximal then the proof is complete. Otherwise there is fuzzy G sub module μ_j with $Supp\mu_j \supseteq Supp\mu_i$. If μ_j is the maximal, the proof is complete. If not, we continue the process and if we cannot find a maximal element in finite steps, we get an ascending chain of fuzzy G sub modules whose corresponding chain of supports is non stationary.

This contradicts our assumption and proves the existence of a maximal fuzzy G sub module for μ.

1) $\Rightarrow$ **2)** Assume that every non empty family of fuzzy G sub modules of μ has a maximal element. Hence there is a fuzzy G sub module μ_i such that, for any fuzzy G sub modules μ_j in the family $Supp\mu_i \supseteq Supp\mu_j$. Consider a chain of fuzzy G sub modules of μ and they must have a maximal element μ_r satisfying $Supp\mu_r \supseteq Supp\mu_j$ for all $j \geq r$. But as the chain is a ascending chain we can conclude that $Supp\mu_r = Supp\mu_j$ for all $j \geq r$. Hence μ satisfies the ascending chain condition.

4.7 Proposition

If a fuzzy G sub module μ on a G module M satisfies ascending chain condition then all its fuzzy G sub modules and quotient fuzzy G sub modules satisfy the a.c.c.

Proof:
Assume ϑ is a fuzzy G sub module of μ. Then any ascending chain $\{\vartheta_i\}$ of fuzzy G sub modules of ϑ is a ascending chain of fuzzy G sub modules of μ. Hence $\{supp\ \vartheta_i\}$ satisfies the a.c.c and thereby proves that the fuzzy G sub module ϑ of μ satisfies a.c.c.
Let $\mu_{(\frac{M}{N})}$ is the quotient fuzzy G sub module of μ *on* M/N. Then any ascending chain $\{\mu_{i(\frac{M}{N})}\}$ of fuzzy G sub modules of $\mu_{(\frac{M}{N})}$ corresponds to an ascending chain of fuzzy G sub modules $\{\mu_i\}$ of μ with $supp\ \mu_{i(\frac{M}{N})} = supp\ \mu_i$. As μ satisfies a.c.c, $\{supp\mu_i\}$ satisfies a.c.c of G modules. This proves that $\mu_{(\frac{M}{N})}$ satisfies a.c.c.

4.8 Proposition

Let μ is a fuzzy G module on a G module M and ϑ is a fuzzy G sub module of μ defined on a G sub module N of M. If ϑ and $\mu_{(\frac{M}{N})}$ satisfies a.cc., then μ also satisfies a.c.c.

Proof: Consider an ascending chain $\{\mu_i\}$ of fuzzy G sub modules of μ. Then $\{\mu_i \cap \vartheta\}$ is ascending chain of fuzzy G sub modules of ϑ. As ϑsatisfies a.c.c there is an integer r such that $supp(\mu_r \cap \vartheta) = supp(\mu_{r+1} \cap \vartheta) = supp\ (\mu_{r+2} \cap \vartheta) = \cdots$

Also $\{\mu_i + \vartheta\}_{\{\frac{supp\ (\mu_i+\vartheta)}{N}\}}$ is an ascending chain of fuzzy G sub modules of $\mu_{(\frac{M}{N})}$. As $\mu_{(\frac{M}{N})}$satisfies a.c.c there is an integer s such that $\frac{supp\ (\mu_s+\vartheta)}{N} = \frac{supp\ (\mu_{s+1}+\vartheta)}{N} = \frac{supp\ (\mu_{s+2}+\vartheta)}{N} = \cdots$

Hence we have
1. $\mu_n \leq \mu_{n+1}$for all n,
2. $supp(\mu_n \cap \vartheta) = supp(\mu_{n+1} \cap \vartheta)$ for all $n \geq r$,
3. $\frac{supp\ (\mu_n+\vartheta)}{N} = \frac{supp\ (\mu_{n+1}+\vartheta)}{N}$ for all $n \geq s$,
4. $supp(\mu_n + \vartheta) \cong \frac{supp\ \mu_n}{Supp\ \{\mu_n \cap \vartheta\}}$

Hence $\frac{supp\ \mu_n}{Supp\ \{\mu_n \cap \vartheta\}} = \frac{supp\ (\mu_n+\vartheta)}{N} = \frac{supp\ (\mu_{n+1}+\vartheta)}{N} = \frac{supp\ \mu_{n+1}}{Supp\ \{\mu_{n+1} \cap \vartheta\}}$ and by using 2 we prove that $supp(\mu_n) = supp(\mu_{n+1})$ for all $n > \max(r, s)$. Hence satisfies a.c.c.

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