Fibrewise Soft Topological Spaces

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Abstract: In this work we define and study new concept of fibrewise topological spaces, namely fibrewise soft topological spaces. Also, we introduce the concepts of fibrewise closed soft topological spaces, fibrewise open soft topological spaces, fibrewise soft near compact spaces and fibrewise locally soft near compact spaces.

Keywords: Soft set, soft continuous, fibrewise soft topological spaces, fibrewise closed soft topological spaces, fibrewise open soft topological spaces, fibrewise soft near compact spaces, fibrewise locally soft near compact space.

AMS Subject Classification: 55R70, 54C05, 54C08, 54C10, 06D72.

1. Introduction and Preliminaries

To begin with we work in the category of fibrewise sets over a given set, called the base set. If the base set is denoted by \( B \) then a fibrewise set over \( B \) consists of a set \( H \) together with a function \( P : H \rightarrow B \), called the projection. For each point \( b \) of \( B \) the fibre over \( b \) is the subset \( H_b = P^{-1}(b) \) of \( H \); fibres may be empty since we do not require \( P \) to be surjective, also for each subset \( B' \) of \( B \) we regard \( H_{B'} = P^{-1}(B') \) as a fibrewise set over \( B' \) with the projection determined by \( P \). Molodtsov [16] generalized with the introduction of soft sets the traditional concept of a set in the classical researches. With the introduction of the applications of soft sets [15], the soft set theory has been the research topic and have received attention gradually [5, 13, 17, 18]. The applications of the soft sets are developed so as to develop and consolidate this theory, utilizing these new applications: a uni-int decision-making method was established [8]. Numerous notions of general topology were involved in soft sets and many authors developed theories about soft topological spaces. Shabir and Naz [21] mentioned this term to define soft topological space. After that definition, I. Zorlutuna et al. [25], A. Aygunoglu et al. [7] and Hussain et al. [11] continued to search the properties of soft topological space. They obtained a lot of vital conclusion in soft topological spaces. We studied the connected between fibrewise topological spaces and soft topological space also some related concepts such as fibrewise soft open, fibrewise soft closed, fibrewise soft near compact and fibrewise locally soft near compact. The purpose of this paper is introduced a new class of fibrewise topology called fibrewise soft topological space are introduced and few of their properties are investigated, we built on some of the result in [1,19,22,23].

Definition 1.1. [12] Let \( H \) and \( K \) are fibrewise sets over \( B \), with projections \( P_H : H \rightarrow B \) and \( P_K : K \rightarrow B \), respectively, a function \( \Phi : H \rightarrow K \) is said to be fibrewise if \( P_K \circ \Phi = P_H \), in other words if \( \Phi(H_b) \subseteq K_{P_H(b)} \) for each point \( b \) of \( B \).

Definition 1.2. [16] Let \( U \) be an initial universe and \( E \) be a set of parameters. Let \( P(U) \) denote the power set of \( U \) and \( A \) be a non-empty subset of \( E \). A pair \( (F,A) \) is called a soft set over \( H \) where \( F \) is a mapping given by \( F : A \rightarrow P(U) \). In other words, a soft set over \( U \) is a parameterized family of subset of the universe \( U \). For \( \varepsilon \in A \), \( F(\varepsilon) \) may be considered as the set \( \varepsilon \) approximate elements of the soft set \( (F,A) \).

Note that the set of all soft sets over \( U \) will be denoted by \( S(U) \).

Example 1.3. Suppose that there are six houses in the universe \( U = \{ u_1, u_2, u_3, u_4, u_5, u_6 \} \) under consideration, and that \( E = \{ e_1, e_2, e_3, e_4, e_5 \} \) is a set of decision parameters. The \( e_i \) (i = 1, 2, 3, 4, 5) stand for the parameters “expensive”, “beautiful”, “wooden”, “cheap”, and “in green surroundings”, respectively. Consider the mapping \( F_A \) given by “houses (.)”, \( \varepsilon \) is to be filled in by one of the parameters \( h_i \) (i ∈ E). For instance, \( F_A(e_1) \) means “houses (expensive)”, and its functional value is the set \( \{ u \in U : u \) is an expensive house \}.

Suppose that \( A = \{ e_2, e_3, e_4 \} \subseteq E \) and \( F_A(e_1) = \{ u_1, u_4 \}, F_A(e_2) = \{ u_1, u_2, u_3 \}, F_A(e_3) = \{ u_4, u_2, u_3 \} \).

Then, we can view the soft set \( F_A \) as consisting of the following collection of approximations: \( F_A = \{ (e_1, \{ u_2, u_3 \}), (e_2, U), (e_3, \{ u_1, u_2, u_3 \}) \} \).

Definition 1.4. [9] Let \( F_A \in S(U) \). A soft topology on \( F_A \), denoted by \( \tau \), is a collection of soft subsets of \( F_A \) having following properties:

(a) \( F_A, F_B \in \tau \)
(b) \( \{ F_{A_i} : i \in I \subseteq N \} \subseteq \tau \Longleftrightarrow \bigcup_{i \in I} F_{A_i} \in \tau \)
(c) \( \{ F_{A_i} : 1 \leq i \leq n, n \in N \} \subseteq \tau \Longleftrightarrow \bigcap_{i=1}^{n} F_{A_i} \in \tau \).

The pair \((F_A, \tau)\) is called a soft topological space.

Example 1.5. [9] Let \( U = \{ u_1, u_2, u_3 \}, E = \{ e_1, e_2, e_3 \}, A = \{ e_1, e_2 \} \subseteq E \) and \( F_A = \{ (e_1, \{ u_1, u_3 \}), (e_2, \{ u_2, u_3 \}) \} \).

Then, \( \tau_1 = \{ F_A, F_B \}, \tau_2 = P(F_A) \) and \( \tau_2 = \{ F_A, F_B = \{ (e_1, \{ u_1, u_3 \}), (e_2, \{ u_2, u_3 \}) \} \).
Definition 1.6. [9]

a) Let \((F_A, \tau)\) be a soft topological space. Then, every element of \(\tau\) is called a soft open set. Clearly, \(F_B\) and \(F_A\) are soft open sets.

b) Let \((F_A, \tau)\) be a soft topological space and \(F_B \subseteq F_A\). Then, the soft closure of \(F_B\), denoted \(\overline{F_B}\), is defined as the soft intersection of all soft closed supersets of \(F_B\). Note that \(\overline{F_B}\) is the smallest soft set containing \(F_B\).

c) Let \((F_A, \tau)\) be a soft topological space and \(\alpha \in F_A\). If there is a soft open set \(F_B\) such that \(\alpha \in F_B\) then \(F_B\) is called a soft open neighborhood (or soft neighborhood) of \(\alpha\). The set of all soft neighborhoods of \(\alpha\), denoted \(\mathcal{V}(\alpha)\), is called the family of soft neighborhoods of \(\alpha\); that is, \(\mathcal{V}(\alpha) = \{F_B : F_B \in \tau, \alpha \in F_B\}\).

d) Let \((F_A, \tau)\) be a soft topological space and \(F_B \subseteq F_A\). Then, \(F_B\) is said to be soft closed if the soft set \(\overline{F_B}\) is soft open.

e) Let \((F_A, \tau)\) be a soft topological space and \(F_B \subseteq F_A\). Then, the collection \(\tau_{F_A} = \{F_B : F_B \in \tau, \exists i \in \mathbb{N} \text{ such that } F_B = \bigcap F_{A_i}\}\) is called a soft subspace topology on \(F_B\). Hence, \((F_B, \tau_{F_A})\) is called a soft topological subspace of \((F_A, \tau)\).

Definition 1.7. A soft set \((F, A)\) in a soft topological space \((F_A, \tau)\) is called

a) Soft α-open set [2] if \((F, A) \subseteq \text{int} (cl(int(F, A)))\).

b) Soft pre-open (briefly, soft P-open) set [6] if \((F, A) \subseteq \text{int} (cl(F, A)) \subseteq \text{int} (cl(F, A)) \subseteq (F, A)\).

c) Soft sime-open (briefly, soft S-open) set [10] if \((F, A) \subseteq (cl(int(F, A)) \cap cl(int(F, A))) \subseteq (F, A)\).

d) Soft β-open set [4] if \((F, A) \subseteq (cl(int(F, A))) \cap (cl(int(F, A))) \subseteq (F, A)\).

e) Soft β-open set [6] if \((F, A) \subseteq (cl(int(F, A))) \cap (cl(int(F, A))) \subseteq (F, A)\).

The complement of a soft α-open (resp. Soft S-open, soft P-open, soft b-open and soft β-open) set is called soft α-closed (resp. Soft S-closed, soft P-closed, soft b-closed and soft β-closed) set. The family of all soft open (resp. Soft S-open, soft P-open, soft b-open and soft β-open) sets of \((F_A, \tau)\) are larger than \(\tau\) and closed under forming arbitrary union. We will call these families soft near topology (briefly S. j-topology), where \(j \in \{\alpha, S, P, b, \beta\}\).

Definition 1.8. [20] Let \(H\) and \(K\) be two non-empty sets and \(E\) be the parameter set. Let \(\{f_e : H \to K, e \in E\}\) be a collection of functions. Then a mapping \(f : SE(H, E) \to SE(K, E)\) defined by \(f_e = e f_{e(k)}\) is called a soft mapping, where \(SE(H, E)\) and \(SE(K, E)\) are sets of all soft elements of the soft sets \((H, E)\) and \((K, E)\) respectively.

Definition 1.9. [2, 4, 14, 24] A soft mapping \(\phi : (H, \tau, E) \to (K, \sigma, L)\) is said to be soft near continuous (briefly S. j-continuous) if the inverse image of each soft open set of \(K\) is a soft j-open set in \(H\) where \(j \in \{\alpha, S, P, b, \beta\}\).

Example 1.10. Let \(H = \{h_1, h_2, h_3\}, \ E = \{e_1, e_2\}\) and \(\tau_1 = \{\phi, \ H', \ (F_1, E), \ (F_2, E)\}\) \(\tau_2 = \{\phi, \ H, \ (G_1, E), \ (G_2, E)\}\) be two soft topologies defined on \(H\). \(F_1, E), (F_2, E), (G_1, E)\) and \(G_2, E)\) are soft sets over \(H\), defined as following: \(F_1\) \(\{e_1 = \{h_1, h_2\}, \ F_2 = \{e_2 = \{h_2, h_3\}\}\) \(H, (G_1, E) = \{\phi, \ H', \ (F_1, E), \ (F_2, E)\}\) \(G_1, E) = \{\phi, \ H, \ (G_1, E), \ (G_2, E)\}\). If we get the mapping \(f : (H, \tau_1) \to (H, \tau_2)\) defined as \(f(h_1) = (h_2, h_3), \) then \(f(h_2) = h_3\) and \(f^{-1}(G_1, E) = (F_1, E), \) and \(f^{-1}(G_2, E) = (F_2, E)\), then \(f\) is a soft continuous mapping.

Definition 1.11. [2, 14, 4, 3] A mapping \(\phi : (H, \tau, E) \to (K, \sigma, L)\) is said to be

a) Soft near-open (briefly, S. j-open) map if the image of every soft open set in \(H\) is S.j-open set in \(K\), where \(j \in \{\alpha, S, P, b, \beta\}\).

b) Soft near-closed (briefly, S. j-closed) map if the image of every soft closed set in \(H\) is S.j-closed set in \(K\), where \(j \in \{\alpha, S, P, b, \beta\}\).

Definition 1.12. [14, 4, 24] Let \(\phi : (H, \tau, E) \to (K, \sigma, L)\) be a function. \(\phi\) is called soft near irresolute(briefly, S. j- irresolute) if the inverse image of soft j-open set in \(K\) is soft j-open in \(H\), where \(j \in \{\alpha, S, P, b, \beta\}\).

Proposition 1.13. [2] If \(\phi : H \to K\) is a soft pre-continuous and soft semi-continuous, then \(\phi\) is soft α-continuous.

2. Fibrewise Soft Topological Spaces

In this section, we give a definition of fibrewise soft topology and its related properties.

Definition 2.1. Assume that \((B, D, G)\) is a soft topology space the fibrewise soft near topology space (briefly, F.W.S. j-topological space) on a fibrewise set \(H\) over \(B\) mean any Soft j-topology space on \(H\) for which the projection \(P\) is soft near continuous(briefly, S. j-continuous) where \(j \in \{\alpha, S, P, b, \beta\}\).

Remark 2.2. In F.W.S. topological space we work over at soft topological base space \(B\), say. When \(B\) is a point-space the theory reduces to that of ordinary soft topology. A F.W.S. topological (resp., S. j-topological) space over \(B\) is just a soft topological (resp., S. j-topological) space \(H\) together with a soft continuous (resp., S. j-continuous) projection \(P_{H,B} : (H, \tau, E) \to (B, D, G)\). So the implication between F.W.S. topological spaces and the families of F.W.S. j-topological spaces are given in the following diagram where \(j \in \{\alpha, S, P, b, \beta\}\).
F.W.S. topological space

F.W.S. α-topological space ⇒ F.W.S. S-topological space

F.W.S.P-topological space ⇒ F.W.S.b-topological space

F.W.S.β-topological space

Example 2.3. Let \( H = \{a, b, c, d\} \), \( E = \{e_1, e_2, e_3\} \), \( G = \{g_1, g_2, g_3\} \) (\( H, \tau, E \)) and let \((B, \Omega, K)\) be a F.W.S. topological space. Define \( f : H \rightarrow B \) and \( u : E \rightarrow G \) as \( f(a) = (b) \), \( f(b) = (d) \), \( f(c) = (a) \), \( f(d) = (c) \); 
\( u(e_1) = \{g_2\}, u(e_2) = \{g_3\}, u(e_3) = \{g_2\} \).
\( \tau = \{ f, H, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \}, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \) are soft sets over \((H, \tau, E)\), defined as follows:

\( (F_1, E) = \{(e_1, [a, b]), (e_2, [c]), (e_3, [a, c])\} \)
\( (F_2, E) = \{(e_1, [b]), (e_2, [a, b]), (e_3, [a, b])\} \)
\( (F_3, E) = \{(e_1, [b]), (e_2, [a, c]), (e_3, [a, c])\} \)
\( (F_4, E) = \{(e_1, [b]), (e_2, H), (e_3, [a, d])\} \)
\( (F_5, E) = \{(e_1, [c]), (e_2, [a, c]), (e_3, [a, b])\} \)
\( (F_6, E) = \{(e_1, [c]), (e_2, [c]), (e_3, [c])\} \)
\( (F_7, E) = \{(e_1, [b]), (e_2, [c]), (e_3, [b])\} \)
\( (F_8, E) = \{(e_1, [b]), (e_2, H), (e_3, [a, d])\} \)
\( (F_9, E) = \{(e_1, [c]), (e_2, [a, c]), (e_3, [a, b])\} \)
\( (F_{10}, E) = \{(e_1, [b]), (e_2, [a]), (e_3, [a])\} \)
\( (F_{11}, E) = \{(e_1, [b]), (e_2, [b]), (e_3, [b])\} \)
\( (F_{12}, E) = \{(e_1, [b]), (e_2, H), (e_3, [a, d])\} \)
\( (F_{13}, E) = \{(e_1, [c]), (e_2, [a, c]), (e_3, [a, b])\} \)
\( (F_{14}, E) = \{(e_1, [b]), (e_2, [c]), (e_3, [c])\} \)
\( (F_{15}, E) = \{(e_1, [c]), (e_2, [c]), (e_3, [c])\} \)

\( f(a) = (d), f(b) = (d), f(c) = (a), f(d) = (c) \); 
\( u(e_1) = \{g_2\}, u(e_2) = \{g_2\}, u(e_3) = \{g_2\} \)

Let us consider the F.W.S. topological space \((H, \tau, E)\) given in Example (2.3): that is, \( \tau = \{ f, H, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \}, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \) are soft sets over \((H, \tau, E)\). Define \( f : H \rightarrow B \) and \( u : E \rightarrow G \) as \( f(a) = (b) \), \( f(b) = (d) \), \( f(c) = (a) \), \( f(d) = (c) \); 
\( u(e_1) = \{g_2\}, u(e_2) = \{g_2\}, u(e_3) = \{g_2\} \).
\( \tau = \{ f, H, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \}, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \) are soft sets over \((H, \tau, E)\). Define \( f : H \rightarrow B \) and \( u : E \rightarrow G \) as \( f(a) = (b) \), \( f(b) = (d) \), \( f(c) = (a) \), \( f(d) = (c) \); 
\( u(e_1) = \{g_2\}, u(e_2) = \{g_2\}, u(e_3) = \{g_2\} \).

Example 2.4. Let \( H = \{a, b, c, d\} \), \( E = \{e_1, e_2, e_3\} \), \( G = \{g_1, g_2, g_3\} \) (\( H, \tau, E \)) and let \((B, \Omega, K)\) be a F.W.S. topological space. Define \( f : H \rightarrow B \) and \( u : E \rightarrow G \) as \( f(a) = (b) \), \( f(b) = (d) \), \( f(c) = (a) \), \( f(d) = (c) \); 
\( f(a) = (b) \), \( f(b) = (d) \), \( f(c) = (a) \), \( f(d) = (c) \);
\( u(e_1) = \{g_2\}, u(e_2) = \{g_1\}, u(e_3) = \{g_2\} \). Let us consider the F.W.S. topological space \((H, \tau, E)\) over \((B, \Omega, G)\) given in Example (2.3): that is, \( \tau = \{ f, H, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \}, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \) are soft sets over \((H, \tau, E)\).

Example 2.5. Let \( H = \{a, b, c, d\} \), \( E = \{e_1, e_2, e_3\} \), and \( G = \{g_1, g_2, g_3\} \) (\( H, \tau, E \)) and let \((B, \Omega, K)\) be a F.W.S. topological space. Define \( f : H \rightarrow B \) and \( u : E \rightarrow G \) as \( f(a) = (b) \), \( f(b) = (d) \), \( f(c) = (a) \), \( f(d) = (c) \); 
\( u(e_1) = \{g_2\}, u(e_2) = \{g_1\}, u(e_3) = \{g_2\} \). Let us consider the F.W.S. topological space \((H, \tau, E)\) over \((B, \Omega, G)\) given in Example (2.3): that is, \( \tau = \{ f, H, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \}, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \) are soft sets over \((H, \tau, E)\). Define \( f : H \rightarrow B \) and \( u : E \rightarrow G \) as \( f(a) = (b) \), \( f(b) = (d) \), \( f(c) = (a) \), \( f(d) = (c) \); 
\( u(e_1) = \{g_2\}, u(e_2) = \{g_1\}, u(e_3) = \{g_2\} \). Let us consider the F.W.S. topological space \((H, \tau, E)\) over \((B, \Omega, G)\) given in Example (2.3): that is, \( \tau = \{ f, H, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \}, (F_1, E), (F_2, E), \ldots, (F_{15}, E) \) are soft sets over \((H, \tau, E)\).
(g_2, [a, b, d]) and let the projection \( P_{U^1} : (H, \tau, E) \to (B, \Omega, G) \) be a soft mapping. Then \((N, G)\) is a soft open in \((B, \Omega, G)\) and \(P_{U^1}^{-1}(N, G) = \{(e_2, [a, c]), (e_1, [d, b]), (e_3, [c, a]), (e_4, [b, d])\}\) is a soft b-open but not soft p-open in \((H, \tau, E)\). Therefore, \(P_{U^1}\) is a soft b-continuous but not soft p-continuous. Thus, \((H, \tau, E)\) is F.W.S. b-topological space but not F.W.S. p-topological space.

Example 2.8. Let \( H = B = \{a, b, c, d\}\), \( E = \{e_1, e_2, e_3\}\), \( G = \{g_1, g_2, g_3\}\). \((H, \tau, E)\) and let \((B, \Omega, G)\) be a F.W.S topological space. Define \( f : H \to B \) and \( u : E \to G \) as \( f(a) = [b], f(c) = [a], f(d) = [c] \). \( u(e_1) = g_2, u(e_2) = g_3, u(e_3) = g_1\). Let us consider the F.W.S. topological space \((H, \tau, E)\) over \((B, \Omega, G)\) given in Example (2.3); that is, \( \tau = \{\phi, H, (F_2, E), (F_3, E), \ldots, (F_{15}, E)\}, \Omega = \{\phi, B, (L, G)\}, \) and \( (L, G) = \{(g_1, [b, d]), (g_2, [a, c]), (g_3, [a, b, d])\}\) and let the projection \( P_{U^1} : (H, \tau, E) \to (B, \Omega, G) \) be a soft mapping. Then \((L, G)\) is a soft open in \((B, \Omega, G)\) and \(P_{U^1}^{-1}(L, G) = \{(e_2, [a, b]), (e_3, [c, d]), (e_4, [a, c, d])\}\) is a soft \( \beta \)-open but not soft b-open in \((H, \tau, E)\). Therefore, \(P_{U^1}\) is a soft \( \beta \)-continuous but not soft b-continuous. Thus, \((H, \tau, E)\) is F.W.S. \( \beta \)-topological space but not F.W.S. b-topological space.

Proposition 2.9. F.W.S. \( s \)-topological space and F.W.S. p-topological space iff F.W.S. \( \alpha \)-topological space.

Proof. \( \iff \) Let \((H, \tau, E)\) be a F.W.S. \( s \)-topological space over \((B, \Omega, G)\), and be a F.W.S. p-topological space over \((B, \Omega, G)\) then the projection \( P_{U^1} : (H, \tau, E) \to (B, \Omega, G) \) exists. To show that \( P_{U^1}\) is soft-continuous. Since \((H, \tau, E)\) is F.W.S.s-topological space over \((B, \Omega, G)\) and \((H, \tau, E)\) is a F.W.S. p-topological space over \((B, \Omega, G)\), then \( P_{U^1}\) is soft \( s \)-continuous and soft p-continuous then \( P_{U^1}\) is soft \( \alpha \)-continuous by proposition (1.15). Thus, \((H, \tau, E)\) is F.W.S. \( \alpha \)-topological space over \((B, \Omega, G)\).

\( \iff \) It obvious.

Let \( \Phi : (H) \to (K, \sigma, L) \) be a fibrewise soft function, \((H)\) is a fibrewise set and \((K, \sigma, L)\) is a fibrewise topological space over \((B, \Omega, G)\). We can give \((H, \tau, E)\) the induced (resp. j-induced) soft topology, in the ordinary sense, and this is necessarily a F.W.S. topology (resp. j-topology). We may refer to it, as the induced (resp. j-induced) F.W.S. topology, where \( j \in \{a, s, P, b, \beta\} \), and note the following characterizations.

**Proposition 2.10.** Let \( \Phi : (H, \tau, E) \to (K, \sigma, L) \) be a fibrewise soft function, \((K, \sigma, L)\) a F.W.S. topological space over \((B, \Omega, G)\) and \((H, \tau, E)\) has the induced F.W.S. topology. Then for each F.W.S. topological space \((Z, \gamma, M)\), a fibrewise soft function \( \Psi : (Z, \gamma, M) \to (H, \tau, E) \) is soft \( j \)-continuous iff the composition \( \Phi \circ \Psi : (Z, \gamma, M) \to (K, \sigma, L) \) is soft \( j \)-continuous, where \( j \in \{a, s, P, b, \beta\} \).

**Proof.** \( \iff \) Suppose that \( \Psi \) is soft \( j \)-continuous. Let \( z \in Z_b \), where \( b \in B \) and \((N, L)\) soft open set of \((\Phi \circ \Psi)(z) = k \in K_b \) in \((K, \sigma, L)\). Since \( \Phi \) is soft continuous, \( \Phi^{-1}(N, L)\) is a soft open set containing \( \Psi(z) = h \in H \) in \((H, \tau, E)\). Since \( \Psi \) is soft \( j \)-continuous, then \( \Psi^{-1}(\Phi^{-1}(N, L)) = \Phi^{-1}(\Psi^{-1}(N, L)) \) is a soft j-open set containing \( z \in Z_b \) in \((Z, \gamma, M)\) and \( \Psi^{-1}(\Phi^{-1}(N, L)) = (\Phi \circ \Psi)^{-1}(N, L) \) is a soft j-open set containing \( z \in Z_b \) in \((Z, \gamma, M)\), where \( j \in \{a, s, P, b, \beta\} \).

**Proposition 2.11.** Let \( \Phi : (H, \tau, E) \to (K, \sigma, L) \) be a fibrewise soft function, \((K, \sigma, L)\) a fibrewise soft topological space over \((B, \Omega, G)\) and \((H, \tau, E)\) has the induced fibrewise soft topology. If for each F.W.S. topological space \((Z, \gamma, M)\), a fibrewise soft function \( \Psi : (Z, \gamma, M) \to (H, \tau, E) \) is soft \( j \)-irresolute iff the composition \( \Phi \circ \Psi : (Z, \gamma, M) \to (K, \sigma, L) \) is soft \( j \)-continuous, where \( j \in \{a, s, P, b, \beta\} \).

**Proof.** The proof is similar to the proof of Proposition (2.10).

**Proposition 2.12.** Let \( \Phi : (H, \tau, E) \to (K, \sigma, L) \) be a fibrewise soft function, \((K, \sigma, L)\) a fibrewise soft topological space over \((B, \Omega, G)\) and \((H, \tau, E)\) has the induced fibrewise soft topology. If for each fibrewise soft topological space \((Z, \gamma, M)\), a fibrewise soft function \( \Psi : (Z, \gamma, M) \to (H, \tau, E) \) is soft open, surjective iff the composition \( \Phi \circ \Psi : (Z, \gamma, M) \to (K, \sigma, L) \) is soft open.

**Proof.** The proof is similar to the proof of Proposition (2.16).

### 3. Fibrewise Soft Near Closed and Soft Near Open Topological Spaces

In this section, we introduce the concepts of fibrewise soft near closed, soft near open topological spaces. Several topological properties on the obtained concepts are studied.

**Definition 3.1.** A F.W.S. topological space \((H, \tau, E)\) over \((B, \Omega, G)\) is called fibrewise soft j-closed (briefly, F.W.S. j-closed ) if the projection \( P_{U^1}\) is soft j-closed, where \( j \in \{a, s, P, b, \beta\} \).

**Proposition 3.2.** Let \( \Phi : (H, \tau, E) \to (K, \sigma, L) \) be a closed fibrewise soft function, where \((H, \tau, E)\) and \((K, \sigma, L)\) are F.W.S. topological spaces over \((B, \Omega, G)\). If \((K, \sigma, L)\) is F.W.S. j-closed, then \((H, \tau, E)\) is F.W.S. j-closed, where \( j \in \{a, s, P, b, \beta\} \).
Proof. Suppose that $\phi : (H, \tau, E) \to (K, \sigma, L)$ is closed fibrewise soft function and $(K, \sigma, L)$ is F.W.S. j-closed i.e., the projection $P_{\pi(K)} : (K, \sigma, L) \to (B, \Omega, G)$ is soft j-closed. To show that $(H, \tau, E)$ is F.W.S. j-closed i.e., the projection $P_{\phi(fa)} : (H, \tau, E) \to (B, \Omega, G)$ is soft j-closed. Now let $(F, C)$ be a soft closed subset of $H_b$, where $b \in B$, since $\phi$ is soft closed, then $(F, C)$ is closed subset of $K_b$. Since $P_{\phi(fa)}$ is soft j-closed, then $P_{\phi(fa)}(F, C)$ is soft j-closed in $(B, \Omega, G)$. Thus, $P_{\phi(fa)}$ is soft j-closed and $(H, \tau, E)$ is F.W.S. j-closed, where $j \in \{a, S, P, b, \beta\}$.

**Proposition 3.3.** Let $(H, \tau, E)$ be a F.W.S topological space over $(B, \Omega, G)$. Suppose that $(H_i, E_i)$ is F.W.S. j-closed for each member $(H_i, E_i)$ of a finite covering of $(H, \tau, E)$. Then $(H, \tau, E)$ is F.W.S. j-closed.

Let $(H, \tau, E)$ be a F.W.S. topological space over $(B, \Omega, G)$, then the projection $P_{H_b} : (H, \tau, E) \to (B, \Omega, G)$ exists. To show that $P_{H_b}$ is soft j-closed. Now, since $(H_i, E_i)$ is F.W.S. j-closed, then the projection $P_{H_b} : (H_i, E_i) \to (B, \Omega, G)$ is soft j-closed for each member $(H_i, E_i)$ of a finite covering of $(H, \tau, E)$. Let $(F, C)$ be a soft j-closed subset of $(H, \tau, E)$, then $P_{H_b}(F, C) = \bigcup (H_i, E_i)(F, C)$ which is a finite union of soft closed sets and hence $P_{H_b}$ is soft j-closed. Thus, $(H, \tau, E)$ is F.W.S. j-closed, where $j \in \{a, S, P, b, \beta\}$.

**Proposition 3.4.** Let $(H, \tau, E)$ be a F.W.S topological space over $(B, \Omega, G)$. Then $(H, \tau, E)$ is F.W.S. j-closed iff each fibre soft set $(H_b, E_b)$ of $(H, \tau, E)$ and each soft open set $(F, E)$ of $(H, \tau, E)$, there exists a soft j-open set $(F, E)$ of $b$ such that $(H_{(F,E),E_{(F,E)}}) \subseteq (F, E)$, where $j \in \{a, S, P, b, \beta\}$.

**Proof:** $(\Rightarrow)$ Suppose that $(H, \tau, E)$ is F.W.S. j-closed i.e., the projection $P_{H_b} : (H, \tau, E) \to (B, \Omega, G)$ is soft j-closed. Now, let $b \in B$ and $(F, E)$ soft open set of $(H_b, E_b)$ in $(H, \tau, E)$, then $(H_b, E_b) \subseteq (F, E)$ is soft closed in $(H, \tau, E)$, this implies $P_{H_b}((H_b, E_b) \subseteq (F, E))$ is soft j-closed in $(B, \Omega, G)$, let $(F, G) = (B, \Omega, G)P_{H_b}((H_b, E_b) \subseteq (F, E))$, then $(F, G)$ a soft j-open set of $(F, E)$ in $(B, \Omega, G)$.

$(\Leftarrow)$ Suppose that the assumption hold and $P_{H_b} : (H, \tau, E) \to (B, \Omega, G)$. Now, let $(F, C)$ be a soft closed subset of $(H, \tau, E)$ and $b \in B \cap P_{H_b}(F, C)$ and each soft open set $(F, E)$ of fibre soft $(H_b, E_b)$ in $(H, \tau, E)$, by assumption there exists a soft j-open set $(F, G)$ of $b$ such that $(H_{(F,G),E_{(F,G)}}) \subseteq (F, E)$, it is easy to show that $(F, G) \subseteq (B, \Omega, G) \cap P_{H_b}(F, C)$, hence $(B, \Omega, G) \cap P_{H_b}(F, C)$ is soft j-open in $(B, \Omega, G)$ and this implies $P_{H_b}(F, L)$ is soft j-closed in $(B, \Omega, G)$ and $P_{H_b}$ is soft j-closed. Thus, $(H, \tau, E)$ is F.W.S. j-closed, where $j \in \{a, S, P, b, \beta\}$.

**Definition 3.5.** A F.W.S $(H, \tau, E)$ over $(B, \Omega, G)$ is called fibrewise soft near open (briefly, F.W.S. j-open) if the projection $P_{H_b}$ is soft j-open where $j \in \{a, S, P, b, \beta\}$.

**Proposition 3.6.** Let $\phi : (H, \tau, E) \to (K, \sigma, L)$ be a soft open fibrewise function, where $(H, \tau, E)$ and $(K, \sigma, L)$ are F.W.S. topological spaces over $(B, \Omega, G)$. If $(K, \sigma, L)$ is F.W.S. j-open, then $(H, \tau, E)$ is F.W.S. j-open, where $j \in \{a, S, P, b, \beta\}$.

**Proof:** Suppose that $\phi : (H, \tau, E) \to (K, \sigma, L)$ is open fibrewise soft function and $(K, \sigma, L)$ is F.W.S. j-open i.e., the projection $P_{\pi(K)} : (K, \sigma, L) \to (B, \Omega, G)$ is soft j-open. To show that $(H, \tau, E)$ is F.W.S. j-open i.e., the projection $P_{\phi(fa)} : (H, \tau, E) \to (B, \Omega, G)$ is soft j-open. Now let $(F, E)$ is soft open subset of $H_b$, where $b \in B$, since $\phi$ is soft open, then $\phi(F, E)$ is soft open subset of $K_b$, since $P_{\phi(fa)}$ is soft j-open, then $P_{\phi(fa)}(\phi(F, E))$ is soft j-open in $(B, \Omega, G)$, but $P_{\phi(fa)}(\phi(F, E)) = (P_{\phi(fa)} \circ \phi)(F, E)$ is soft j-open in $(B, \Omega, G)$. Thus, $P_{\phi(fa)}$ is soft j-open and $(H, \tau, E)$ is F.W.S. j-open, where $j \in \{a, S, P, b, \beta\}$.

**Proposition 3.7.** Let $(H_{r}, \tau_{r}, E_{r})$ be a finite family of F.W.S j-open spaces over $(B, \Omega, G)$. Then the F.W.S. topological product $(H_{r}, \tau_{r}, E_{r})$ is also F.W.S. j-open, where $j \in \{a, S, P, b, \beta\}$.

**Proof:** Let $(H_{r}, \tau_{r}, E_{r})$ be a finite family of F.W.S j-open spaces. Suppose that $(H_{r}, \tau_{r}, E_{r}) \subseteq \prod_{r}(H_{r}, \tau_{r}, E_{r})$ is a F.W.S over $(B, \Omega, G)$, then the projection $P_{\phi_{f(a)}} : (H_{r}, \tau_{r}, E_{r}) \to (B, \Omega, G)$ is exists. To show that $P_{\phi_{f(a)}}$ is soft j-open. Now, since $(H_{r}, \tau_{r}, E_{r})$ be a finite family of F.W.S j-open spaces over $(B, \Omega, G)$, the projection $P_{\phi_{f(a)}}(H_{r}, \tau_{r}, E_{r}) \subseteq (B, \Omega, G)$ is soft j-open for each $r$. Let $(F, E)$ be a soft open subset of $(H_{r}, \tau_{r}, E_{r})$, then $P_{\phi_{f(a)}}(F, E)$.

**Remark 3.8.** If $(H, \tau, E)$ is F.W.S. open (resp. F.W.S. j-open) then the second projection $\pi_{2} : (H, \tau, E) \times_{B}(K, \sigma, L) \to (K, \sigma, L)$ is soft open (resp. Soft j-open) for all F.W.S. topological space $(K, \sigma, L)$. Because for every nonempty soft open (resp. Soft open, Soft j-open and Soft j-open) set $(F, E)$, we have...
\[ \pi_2((F, E) \times_B (F, L)) = (F, L) \] is soft open (resp. Soft j-open, Soft open and Soft j-open), where \( j \in \{a, S, P, b, \beta\} \) We will use this in the proof of the following results.

**Proposition 3.9.** Let \( \phi : (H, \tau, E) \rightarrow (K, \sigma, L) \) be a fibrewise soft function, where \((H, \tau, E)\) and \((K, \sigma, L)\) are F.W.S. topological spaces over \((B, \Omega, G)\). Let \( i\pi_2 \times \phi : (H, \tau, E) \times_B (K, \sigma, L) \rightarrow (H, \tau, E) \times_B (K, \sigma, L) \) be soft open and that \((H, \tau, E) \times_B (K, \sigma, L)\) is F.W.S. open, \((K, \sigma, L)\) is F.W.S. j-open. Then \( \phi \) itself is j-open, where \( j \in \{a, S, P, b, \beta\} \).

**Proof.** Consider the following commutative figure.

\[
\begin{array}{ccc}
H \times_B H & \xrightarrow{i\pi_2 \times \phi} & H \times_B K \\
\downarrow & & \downarrow \\
\{1\} \times_B H & \xrightarrow{\phi} & \{1\} \times_B K
\end{array}
\]

**Figure 1:** Diagram of Proposition 3.9

The projection on the left is surjective and soft j-open, since \((K, \sigma, L)\) is F.W.S. j-open, while the projection on the right is soft j-open, since \((H, \tau, E)\) is F.W.S. j-open. Therefore, \( \pi_2 \circ (i\pi_2 \times \phi) = \phi \circ \pi_2 \) is soft j-open, and so \( \phi \) is soft j-open, by Proposition (2.3) as asserted, where \( j \in \{a, S, P, b, \beta\} \).

**Proposition 3.10.** Let \( \phi : (H, \tau, E) \rightarrow (K, \sigma, L) \) be a soft j-continuous fibrewise surjection, where \((H, \tau, E)\) and \((K, \sigma, L)\) are F.W.S. topological spaces over \((B, \Omega, G)\).

**Proof.** Suppose that \( \phi : (H, \tau, E) \rightarrow (K, \sigma, L) \) is soft j-continuous fibrewise surjection and \((H, \tau, E)\) is F.W.S. j-closed (resp. F.W.S. j-open) i.e., the projection \( P_{\phi} : (H, \tau, E) \rightarrow (B, \Omega, G) \) is soft j-closed (resp. Soft j-open). To show that \((K, \sigma, L)\) is F.W.S. j-closed (resp. F.W.S. j-open) over \((B, \Omega, G)\).

**4. Fibrewise Soft Near Compact and Locally Soft Near Compact Spaces.**

In this section, we study fibrewise soft near compact and fibrewise locally soft near compact spaces as a generalizations of well-known concepts soft near compact and locally soft near compact topological spaces.

**Definition 4.1.** The function \( \phi : (H, \tau, E) \rightarrow (K, \sigma, L) \) is called soft near proper (briefly S. j-proper) function if it is S. j-continuous, closed and for each \( (k) \in (K, \phi^{-1}(k)) \) is compact set, where \( j \in \{a, S, P, b, \beta\} \).

For example, let \( (R, \tau, E)\) be the topology with basis whose members are of the form \( (a, b) \) and \( (a, b) = N, N = \{1 \in \mathbb{N} : n \in \mathbb{Z}^+ \} \) and \( E = N \) Define \( \phi : (R, \tau, E) \rightarrow (R, \sigma, L) \) by \( \phi ((F, E)) = (F, E) \), then \( \phi \) is soft j-proper function, where \( j \in \{a, S, P, b, \beta\} \).

If \( \phi : (H, \tau, E) \rightarrow (K, \sigma, L) \) is fibrewise and S. j-proper function, then \( \phi \) is said to be fibrewise S. j-proper function, where \( j \in \{a, S, P, b, \beta\} \).

**Definition 4.2.** The F.W.S. topological space \((H, \tau, E)\) over \((B, \Omega, G)\) is called fibrewise soft j-compact (briefly F.W.S. j-compact) if the projection \( P_{\phi} : (H, \tau, E) \rightarrow (B, \Omega, G) \) is soft j-proper, where \( j \in \{a, S, P, b, \beta\} \).

**Proposition 4.3.** The F.W.S. topological space \((H, \tau, E)\) over \((B, \Omega, G)\) is F.W.S. j-compact iff \((H, \tau, E)\) is fibrewise soft closed and every fibre of \((H, \tau, E)\) is S. j-compact, where \( j \in \{a, S, P, b, \beta\} \).

**Proof.** Let \((H, \tau, E)\) be a F.W.S. j-compact space, then the projection \( P_{\phi} : (H, \tau, E) \rightarrow (B, \Omega, G) \) is soft j-proper function i.e., \( P_{\phi} \) is soft closed and for each \( b \in B, H_b \) is soft...
Let \((\mathcal{H}, \tau, E)\) be F.W.S. closed and every fibre soft of \((\mathcal{H}, \tau, E)\) is soft j-compact, where \(j \in \{a, S, P, b, \beta\}\).

\(\iff\) Let \((\mathcal{H}, \tau, E)\) be F.W.S. closed and every fibre soft of \((\mathcal{H}, \tau, E)\) is soft j-compact, then the projection \(P_{(f,a)}: (\mathcal{H}, \tau, E) \rightarrow (b, \Omega, G)\) is closed and it is clear that \(P_{(f,a)}\) is S-j-continuous, also for each \(b \in B\), \(H_b\) is soft j-compact. Hence \((\mathcal{H}, \tau, E)\) is F.W.S. j-compact, where \(j \in \{a, S, P, b, \beta\}\).

**Proposition 4.4.** Let \((\mathcal{H}, \tau, E)\) be a F.W.S. topological space over \((B, \Omega, G)\). Then is F.W.S. j-compact iff for each fibre soft \(H_z\) of \((\mathcal{H}, \tau, E)\) and each covering \(\pi_z\) of \(H_z\) by soft open sets of \(H\) there exists a soft nbd \((N, G)\) of \(b\) such that a finite subfamily of \(\pi_z\) covers \(H_{(\mathcal{H}, \tau, E)}\) where \(j \in \{a, S, P, b, \beta\}\).

**Proof.** \((\Rightarrow)\) Let \((\mathcal{H}, \tau, E)\) be F.W.S. j-compact space, then the projection \(P_{(f,a)}: (\mathcal{H}, \tau, E) \rightarrow (B, \Omega, G)\) is soft j-proper function, so that \(H_z\) is soft j-compact for each \(b \in B\). Let \(\pi_z\) be a covering of \(H_z\) by soft open sets \(H\) for each \(b \in B\) and let \(H_{(\mathcal{H}, \tau, E)}\) be unique soft j-compact sets of \(H\) there exists a soft nbd \((N, G)\) of \(b\) such that a finite subfamily of \(\pi_z\) covering \(H_{(\mathcal{H}, \tau, E)}\) where \(j \in \{a, S, P, b, \beta\}\).

**Proposition 4.5.** Let \(\phi: (\mathcal{H}, \tau, E) \rightarrow (k, \sigma, L)\) be a j-proper, j-closed fibrewise soft function, where \((\mathcal{H}, \tau, E)\) and \((k, \sigma, L)\) are F.W.S. topological spaces over \((B, \Omega, G)\). If \((k, \sigma, L)\) is F.W.S. j-compact then so is \((\mathcal{H}, \tau, E)\), where \(j \in \{a, S, P, b, \beta\}\).

**Proof:** Suppose \(\phi: (\mathcal{H}, \tau, E) \rightarrow (k, \sigma, L)\) is j-proper, j-closed fibrewise soft function and \((k, \sigma, L)\) is F.W.S. j-compact space i.e., the projection \(P_{(q, d)}: (k, \sigma, L) \rightarrow (B, \Omega, G)\) is j-proper. To show that \((\mathcal{H}, \tau, E)\) is F.W.S. j-compact space i.e., the projection \(P_{(f,a)}: (\mathcal{H}, \tau, E) \rightarrow (B, \Omega, G)\) is soft j-proper. Now, clear that \(P_{(f,a)}\) is soft j-continuous. Let \((\mathcal{F}, L)\) be a soft closed subset of \(H_b\) where \(b \in B\). Since \(\phi\) is closed, then \(\phi(\mathcal{F}, L)\) is soft closed subset of \(K_b\). Since \(P_{(q, d)}\) is soft closed, then \(P_{(f,a)}(\phi(\mathcal{F}, L))\) is soft closed in \((B, \Omega, G)\). But \(P_{(q, d)}(\phi(\mathcal{F}, L)) = (P_{(q, d)} \circ \phi)(\mathcal{F}, L)\) so that \(P_{(f,a)}(\phi(\mathcal{F}, L))\) is soft closed in \((B, \Omega, G)\). Thus, \(P_{(f,a)}(\phi(\mathcal{F}, L))\) is soft closed in \((B, \Omega, G)\).

A similar result holds for finite coproducts.
is soft j-continuous. Let $H_f = \bigcup H_j$ which is a finite union of soft j-compact sets and hence $H_f$ is soft j-compact. Thus, $P_{f_u}$ is soft j-proper and $(H, \tau, E)$ is F.W.S. j-compact, where $j \in \{a, \alpha, P, b, \beta\}$.

**Definition 4.8.** A F.W.S. topological space $(H, \tau, E)$ is called fibrewise soft j-irresolute (briefly, F.W.S. j-irresolute ) if the projection $P_{f_u}$ is soft j-irresolute, where $j \in \{a, \alpha, P, b, \beta\}$.

**Proposition 4.9.** Let $\phi : (H, \tau, E) \rightarrow (K, \sigma, L)$ be a soft j-continuous, soft j-irresolute fibrewise surjection, where $(H, \tau, E)$ and $(K, \sigma, L)$ are F.W.S. topological spaces over $(B, \Omega, G)$. If $(H, \tau, E)$ is F.W.S. j-compact then so is $(K, \sigma, L)$, where $j \in \{a, \alpha, P, b, \beta\}$.

**Proof.** Suppose that $\phi : (H, \tau, E) \rightarrow (K, \sigma, L)$ is a soft j-continuous, soft j-irresolute fibrewise surjection and $(H, \tau, E)$ is F.W.S. j-compact i.e., the projection $P_{f_u}$ is soft j-irresolute, where $j \in \{a, \alpha, P, b, \beta\}$. Suppose that $P_{f_u}$ is soft j-proper. Now, it is clear that $P_{f_u}$ is soft j-continuous. Since $(H, \tau, E)$ is F.W.S. j-compact over $(B, \Omega, G)$, then the projection $P_{f_u}$ is soft j-continuous over $(B, \Omega, G)$, where $j \in \{a, \alpha, P, b, \beta\}$.

In fact the last result is also holds for locally finite soft j-closed coverings, instead of soft j-open coverings.

**Proposition 4.10.** Let $(H, \tau, E)$ be F.W.S. j-compact space over $(B, \Omega, G)$. Then $(H^*_\alpha, \tau^*_\alpha, E^*_\alpha)$ is F.W.S. j-compact space over $(B^*, \Omega^*, G^*)$ for each soft subspace $(B^*, \Omega^*, G^*)$ of $(B, \Omega, G)$, where $j \in \{a, \alpha, P, b, \beta\}$.

**Proof.** Suppose that $(H, \tau, E)$ is F.W.S. j-compact i.e., the projection $P_{f_u} : (H, \tau, E) \rightarrow (B, \Omega, G)$ is soft j-proper. To show that $(H^*_\alpha, \tau^*_\alpha, E^*_\alpha)$ is F.W.S. j-compact space over $(B^*, \Omega^*, G^*)$, i.e., the projection $P_{f_u} : (H^*_\alpha, \tau^*_\alpha, E^*_\alpha) \rightarrow (B^*, \Omega^*, G^*)$ is soft j-proper. Now, it is clear that $P_{f_u}$ is soft j-continuous. Let $(F, L)$ be a soft closed subset of $(H, \tau, E)$, then $(F, L) \cap (H^*_\alpha, \tau^*_\alpha, E^*_\alpha)$ is soft closed in $(H^*_\alpha, \tau^*_\alpha, E^*_\alpha)$ and hence $P_{f_u}(F, L) \cap (B^*, \Omega^*, G^*)$ is soft closed in $(B^*, \Omega^*, G^*)$. Thus, $P_{f_u}$ is soft j-continuous and $(H, \tau, E)$ is F.W.S. j-compact, where $j \in \{a, \alpha, P, b, \beta\}$.

**Proposition 4.11.** Let $(H, \tau, E)$ be F.W.S. topological space over $(B, \Omega, G)$. Suppose that $(H_{B_1}, E_{B_1})$ is F.W.S. j-compact space over $(B_1, \Omega_1, G_1)$ for each member $(B_1, \Omega_1, G_1)$ of a soft j-open covering of $(B, \Omega, G)$ Then $(H, \tau, E)$ is F.W.S. j-compact over $(B, \Omega, G)$, where $j \in \{a, \alpha, P, b, \beta\}$.

**Proof.** Suppose that $(H, \tau, E)$ is F.W.S. topological space over $(B, \Omega, G)$, then the projection $P_{f_u} : (H, \tau, E) \rightarrow (B, \Omega, G)$ exist. To show that $P_{f_u}$ is soft j-proper. Now, it is clear that $P_{f_u}$ is soft j-continuous. Since $(H_{B_1}, E_{B_1})$ is F.W.S. j-compact over $(B_1, \Omega_1, G_1)$, then the projection $P_{f_u} : (H_{B_1}, E_{B_1}) \rightarrow (B_1, \Omega_1, G_1)$ is soft j-proper for each member $(B_1, \Omega_1, G_1)$ of a soft j-open covering of $(B, \Omega, G)$, Let $(F, L)$ be a soft closed sub-set of $(H, \tau, E)$, then $P_{f_u}(F, L) \cap (B_{B_1, E_{B_1}})$ which is a union of soft closed sets and hence $P_{f_u}$ is soft closed. So $P_{f_u}$ is soft j-proper. Let $b \in B_1$ then $H_f = \bigcup_{b \in B_1} (H_{B_1})_{b}$ for every $b \in B_1$. Since $(H_{B_1}, E_{B_1})$ is soft j-compact in $(H, \tau, E)$ and the union of soft j-compact sets is soft j-compact over $(B, \Omega, G)$, where $j \in \{a, \alpha, P, b, \beta\}$.
b ∈ B, there exists a soft nbhd \((N, G)\) of b and an open set \((F, E) \subset H(N, G)\) of \(h\) such that the closure of \((F, E)\) in \(H(N, G)\) (i.e., \(\overline{H(N, G)} \cap C(F, E)\)) is F.W.S. j-compact over \((N, G)\), where \(j \in \{α, S, P, b, β\}\).

Remark 4.14. F.W.S j-compact spaces are necessarily F.W.L.S. j-compact by taken \(H = B\) and \(H' = H\). But the conversely is not true for example, let \(\left( H, \tau_{dis}^E, E \right)\) where \(H\) and \(E\) is infinite set and \(τ_{dis}\) is discrete soft topology, then \(\left( H, \tau_{dis}^E, E \right)\) F.W.L.S. j-compact over \((R, Ω, G)\), since for each \(h \in H\), where \(b \in B\), there exists a soft nbhd \((N, G)\) of \(b\) and an open \((F, E) \subset H(N, G)\) of \((H, \tau_{dis}^E, E)\) such that \(\overline{C(F, E)} \cap H(N, G)\) is F.W.S. j-compact over \((N, G)\), where \(j \in \{α, S, P, b, β\}\). But \(\left( H, τ, E \right)\) is not F.W.S. j-compact over \((R, Ω, G)\), where \(j \in \{α, S, P, b, β\}\).

Proposition 4.15. Let \(\hat{ϕ} : (H, τ, E) \rightarrow (H^*, τ^*, E^*)\) be a closed fibrewise soft embedding, where \(H, τ, E\) and \(H^*, τ^*, E^*\) are F.W.S. topological spaces over \((B, Ω, G)\). If \((H^*, τ^*, E^*)\) is F.W.S. j-compact then so is \((H, τ, E)\), where \(j \in \{α, S, P, b, β\}\).

Proof. Let \(h \in H\), where \(b \in B\). Since \((H^*, τ^*, E^*)\) is F.W.S. j-compact there exists a soft nbhd \((N, G)\) of \(b\) and an open \((F, E) \subset H(N, G)\) of \(h\) such that the closure \(H(N, G)\) of \((F, E)\) in \((H^*, τ^*, E^*)\) is F.W.S.j-compact over \((N, G)\). Then \(\hat{ϕ}^{-1}(F, E) \subset H(N, G)\) is an open set of \(H\) such that the closure \(H(N, G)\) of \(\hat{ϕ}^{-1}(F, E)\) in \((H^*, τ^*, E^*)\) is F.W.S. j-compact over \((F, E)\). Thus, \((H, τ, E)\) is F.W.S. j-compact, where \(j \in \{α, S, P, b, β\}\).

The class of F.W.L.S. j-compact spaces is finitely multiplicative, where \(j \in \{α, S, P, b, β\}\), in the following sense.

Proposition 4.16. Let \(\left( H_r, τ_r, E_r \right)\) be a finite family of F.W.L.S. j-compact spaces over \((B, Ω, G)\). Then the F.W.S. topological product \((H, τ, E) = \prod_r (H_r, τ_r, E_r)\) is F.W.L.S. j-compact, where \(j \in \{α, S, P, b, β\}\).

Proof. The proof is similar to that of Proposition (4.6).

References
