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Meromorphic Starlike Univalent Functions with Positive Coefficients

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Abstract: ***In this paper we obtained sharp results concerning coefficient estimates, distortion theorem, radius of convexity and closure theorem for the class*** $\sigma_p(\alpha,\beta,\xi)$.

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1. Introduction

Let Σ denote the class the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n \qquad (1.1)$$

which are regular in domain E = {z :0 < |z| < 1} with a simple pole at the origin with residue 1 there.

Let Σ_s, $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ $(0 \le \alpha < 1)$ denote the subclasses of Σ that are univalent, moromorphically starlike of order α and meromorphically convex of order α respectively. Analytically $f(z)$ of the form (1.1) is in $\Sigma^*(\alpha)$ if and only if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\}>\alpha,\ z\in E \qquad (1.2)$$

Similarly, $f \in \Sigma_k(\alpha)$ if and only if, $f(z)$ is of the form (1.1) and satisfies

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha,\ z\in E \qquad (1.3)$$

It being understood that if α = 1 then $f(z)=\frac{1}{z}$ is the only function which is $\Sigma^*(1)$ and $\Sigma_k(1)$.

The classes $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ have been extensively studied by Pommerenke [5], Clunie [1], Royster [6] and others.

Since to a certain extent the work in the meromorphic univalent case has paralleled that of regular univalent case, it is natural to search for a subclass of Σ_s that has properties analogous to those of $T^*(\alpha)$. Juneja and Reddy [3] introduced the class Σ_p of functions of the form (1.1) that are meromorphic and univalent in *E*. They showed that the class

$$\sum\nolimits_p^*(\alpha)=\sum\nolimits_p\cap\sum\nolimits^*(\alpha).$$

Also, Mogra, Reddy and Juneja [4] introduced the class of meromorphically starlike function of order α and type β which is denoted by $\sum_p^*(\alpha,\beta)$ They showed that the class

$$\sum\nolimits_p^*(\alpha,\beta)=\sum\nolimits_p\cap\sum(\alpha,\beta)$$

and extended some of the results of Juneja and Reddy [3] to this class..

The aim of the present paper is to introduce the class $\sigma_p(\alpha,\beta,\xi)$ consisting the functions of the form (1.1) which satisfies the condition

$$\left|\frac{\dfrac{zf'(z)}{f(z)}+1}{2\xi\left(\dfrac{zf'(z)}{f(z)}+\alpha\right)-\left(\dfrac{zf'(z)}{f(z)}+1\right)}\right|<\beta \text{ for } |z|<1.$$

where $0\le\alpha<1$, $0<\beta\le 1$ and $\frac{1}{2}<\xi\le 1$.

We find a necessary and sufficient condition , coefficient inequality, distortion properties and radius of convexity and other properties. The results of this paper is generalize the results of Mogra, Reddy and Juneja [4].

2. Main Results

Definition 2.1: $\sigma_p(\alpha,\beta,\xi)$ denote the subclass of Σ consisting of the functions of the form $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n$ which satisfies

$$\left|\frac{\dfrac{zf'(z)}{f(z)}+1}{2\xi\left(\dfrac{zf'(z)}{f(z)}+\alpha\right)-\left(\dfrac{zf'(z)}{f(z)}+1\right)}\right|<\beta,\ |z|<1.$$

where $0\le\alpha<1$, $0<\beta\le 1$ and $\frac{1}{2}<\xi\le 1$.

3. Coefficient Estimates

The following theorem give a sufficient condition for a function to be in $\sum^*(\alpha,\beta,\xi)$.

Theorem 2.1: Let $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n$ be regular in *E*. if

$$\sum_{n=1}^{\infty}\left[(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1\right]\frac{1}{(n+2)^{\sigma}}(1-\alpha)|a_n|\le 2\beta\xi \quad (2.1)$$

$0\le\alpha<1$, $0<\beta\le 1$, $\sigma>0$ and $\frac{1}{2}<\xi\le 1$, then $f\in\sum^*(\alpha,\beta,\xi,\sigma)$.

Proof: Suppose (2.1) holds for all admissible values of α, β and ξ. Consider the expression

$$H(f,f')=\left|z(I^{\sigma}f(z))'+f(z)\right|-\beta\left|2\xi\left(zf'(z)+\alpha f(z)\right)-\left(zf'(z)+f(z)\right)\right| \quad (2.2)$$

Replacing $f(z)$ and $f'(z)$ by their series expansions, we have for $0<|z|=r<1$.

$$H(f,f')=\left|\sum_{n=1}^{\infty}(n+1)a_n z^n\right|-\beta\left|2\xi(\alpha-1)\frac{1}{z}+\sum_{n=1}^{\infty}(2\xi n+2\xi\alpha-n-1)a_n z^n\right|$$

or

$$rH(f,f')\le\sum_{n=1}^{\infty}(n+1)|a_n|r^{n+1}-\beta\left\{2\xi(1-\alpha)-\sum_{n=1}^{\infty}(2\xi n+2\xi\alpha-n-1)|a_n|r^{n+1}\right\}$$

$$=\sum_{n=1}^{\infty}[(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1]|a_n|r^{n+1}-2\beta\xi(1-\alpha)$$

Since the above inequality holds for all r, $0<r<1$, letting $r\to 1$, we have

$$H(f,f')\le\sum_{n=1}^{\infty}[(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1]|a_n|-2\beta\xi(1-\alpha)$$

$$\le 0,$$

by (2.1). Hence it follows that

$$\left|\frac{zf'(z)}{f(z)}+1\right|<\beta\left|2\xi\left(\frac{zf'(z)}{f(z)}+\alpha\right)-\left(\frac{zf'(z)}{f(z)}+1\right)\right|$$

so that $f\in\sum^*(\alpha,\beta,\xi)$. Hence the theorem.

Theorem 2.2: Let $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n, a_n\ge 0$, be regular in *E*. Then $f(z)\in\sigma_p(\alpha,\beta,\xi)$ if only if (2.1) is satisfied.

Proof: In view of theorem 2.1 it is sufficient to show that 'only if' part. Let us assume that

$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z_n, a_n\ge 0$ is in $\sigma_p(\alpha,\beta,\xi)$ .Then

$$\left|\frac{\frac{zf'(z)}{f(z)}+1}{2\xi\left(\frac{zf'(z)}{f(z)}+\alpha\right)-\left(\frac{zf'(z)}{f(z)}+1\right)}\right|=\left|\frac{\sum_{n=1}^{\infty}(n+1)a_n z^n}{2\xi(1-\alpha)\frac{1}{z}-\sum_{n=1}^{\infty}(2\xi n+2\xi\alpha-n-1)a_n z^n}\right|<\beta$$

for all $z\in E$. Using the fact that $Re(z)\le|z|$ for all z.

It follows that

$$Re\left\{\frac{\sum_{n=1}^{\infty}(n+1)a_n z^n}{2\xi(1-\alpha)\frac{1}{z}-\sum_{n=1}^{\infty}(2\xi\, n+2\,\xi\,\alpha-n-1)a_n z^n}\right\}<\beta,\ z\in E. \quad (2.3)$$

Now choose the values of z on the real axis so that $\frac{zf'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1$. through positive values, we obtain

$$\sum_{n=1}^{\infty}(n+1)a_n \le \beta\left\{2\xi(1-\alpha)-\sum_{n=1}^{\infty}(2\xi n+2\xi\alpha-n-1)a_n\right\}$$

or

$$\sum_{n=1}^{\infty}\left[(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1\right]a_n \le 2\beta\xi(1-\alpha)$$

Hence the result follows.

Corollary 2.1: If $f(z)\in\ \sigma_p(\alpha,\beta,\xi)$ then

$$a_n \le \frac{2\beta\xi(1-\alpha)}{(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1},\quad n=1,2,\ldots\ldots \quad (2.4)$$

with equality for each n, for function of the form

$$f_n(z)=\frac{1}{z}+\frac{2\beta\xi(1-\alpha)}{(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1}z^n \quad (2.5)$$

Remark 2.1: If $f(z)\in\ \sigma_p(\alpha,\beta,1)$ i.e., replacing $\xi = 1$, we obtain

$$a_n \le \frac{2\beta(1-\alpha)}{(1+\beta)n+(2\alpha-1)\beta+1},\quad n=1,2,3,\ldots\ldots$$

Equality holds for

$$f_n(z)=\frac{1}{z}+\frac{2\beta(1-\alpha)}{(1+\beta)n+(2\alpha-1)\beta+1}z^n.$$

which is known result of Mogra, Reddy and Juneja [4].

Remark 2.2: If $f(z)\in\ \sigma_p(\alpha,1,1)$ i.e., replacing $\beta = 1$ and $\xi = 1$. We obtain

$$a_n \le \frac{(1-\alpha)}{n+\alpha},\quad n=1,\ 2,\ 3,\ldots\ldots$$

with equality, for each n, for functions of the form.

$$f_n(z)=\frac{1}{z}+\frac{1-\alpha}{n+\alpha}z^n$$

which is known result of Jeneja and Reddy [3].

4. Distortion Property and Radius of Convexity Estimates

Theorem 2.3: If $f(z)\in\ \sigma_p(\alpha,\beta,\xi)$, then for $0 < |z| = r < 1$

$$\frac{1}{r}-\frac{\beta\xi(1-\alpha)}{1-\beta[1-(1+\alpha)\xi]}r\le|f(z)|\le\frac{1}{r}+\frac{\beta\xi(1-\alpha)}{1-\beta[1-(1+\alpha)\xi]}r$$

(2.6)

where equality holds for the funciton

$$f_1(z)=\frac{1}{z}+\frac{\beta\xi(1-\alpha)}{1-\beta[1-(1+\alpha)\xi]}z.$$ At $z = ir$, r (2.7)

Proof: Suppose $f(z)\in\ \sigma_p(\alpha,\beta,\xi)$. In view of Theorem 2.2

We have

$$\sum_{n=1}^{\infty}a_n \le \frac{\beta\xi(1-\alpha)}{1-\beta[1-(1+\alpha)\xi]} \quad (2.8)$$

Thus for $0 < |z| = r < 1$.

$$|f(z)|=\left|\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n\right|\le\left|\frac{1}{z}\right|+\sum_{n=1}^{\infty}a_n z^n|z|^n$$

$$\le\frac{1}{r}+r\sum_{n=1}^{\infty}a_n$$

$$\le\frac{1}{r}+\frac{\beta\xi(1-\alpha)}{1-\beta[1-(1+\alpha)\xi]}r,$$

by (2.8). This gives the right hand inequality of (2.6).

Also,

$$|f(z)|\ge\frac{1}{r}-\sum_{n=1}^{\infty}a_n r\ge\frac{1}{r}-\frac{\beta\xi(1-\alpha)}{1-\beta[1-(1+\alpha)\xi]}r$$

which gives the left hand side of (2.6).

It can be easily seen that the function $f_1(z)$ defined by (2.7) is extremal for the theorem.

Theorem 2.4: If $f(z)$ is in $\sigma_p(\alpha,\beta,\xi)$, then $f(z)$ is meromorphically convex of order $\delta(0\le\delta<1)$ in $|z|<r=r(\alpha,\beta,\xi,\delta)$, where

$$r(\alpha,\beta,\xi,\delta)=\inf_n\left\{\frac{(1-\delta)[(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1]}{2\beta\xi(1-\alpha)n(n+2-\delta)}\right\}^{(1/n+1)}, \quad n=1,2,.....$$

The bound for $|z|$ is sharp for each n, with the extremal function being of the form (2.5).

Proof: Let $f(z)\in\sigma_p(\alpha,\beta,\xi)$ Then, by Theorem 2.2

$$\sum_{n=1}^{\infty}\frac{[(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1]}{2\beta\xi(1-\alpha)}a_n\le 1. \tag{2.9}$$

It is sufficient to show that

$$\left|2+\frac{zf''(z)}{f'(z)}\right|\le 1-\delta \ \text{ for } |z|<r(\alpha,\beta,\xi,\delta)$$

or equivalently, to show that

$$\left|\frac{f'(z)+(zf'(z))'}{f'(z)}\right|\le 1-\delta \tag{2.10}$$

for $|z|<r(\alpha,\beta,\xi,\delta)$

where $r(\alpha,\beta,\xi,\delta)$ is as specified in the statement of the theorem.

Substituting the series expansions for $f'(z)$ and $(zf'(z))'$ in the left side of (2.10) we have

$$\left|\frac{\sum_{n=1}^{\infty}n(n+1)a_nz^{n-1}}{-\frac{1}{z^2}+\sum_{n=1}^{\infty}na_nz^{n-1}}\right|\le\frac{\sum_{n=1}^{\infty}n(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty}na_n|z|^{n+1}}$$

This will be bounded by ($1-\delta$) if

$$\sum_{n=1}^{\infty}\frac{n(n+2-\delta)}{1-\delta}a_n|z|^{n+1}\le 1. \tag{2.11}$$

In view of (2.9), it follows that (2.11) is true if

$$\frac{n(n+2-\delta)}{1-\delta}|z|^{n+1}\le\frac{(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1}{2\beta\xi(1-\alpha)}, \quad n=1,\ 2,.......\text{or}$$

$$|z|\le\left\{\frac{(1-\delta)[1-\beta+2\beta\xi]n+(2\alpha\xi-1)\beta+1}{2\beta\xi(1-\alpha)n(n+2-\delta)}\right\}^{(1/n+1)} \quad n=1,\ 2,..... \tag{2.12}$$

setting $|z|=r(\alpha,\beta,\xi,\delta)$ in (2.12), the result follows.

The result is sharp, the extremal function being of the form

$$f_n(z)=\frac{1}{z}+\frac{2\beta\xi(1-\alpha)}{(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1}z^n$$

Corollary 2.2: If $f\in\sigma_p(\alpha,\beta,1)$, then f is convex in the disk

$$0<|z|<r(\alpha,\beta,\xi,\delta)=\inf_n\left\{\frac{(1-\delta)[(1+\beta)n+(2\alpha-1)\beta+1]}{2\beta(1-\alpha)n(n+2-\delta)}\right\}^{[1/(n+1)]} \quad n=1,2,3,......$$

The result is sharp for

$$f_n(z)=\frac{1}{z}+\frac{2\beta(1-\alpha)}{(1+\beta)n+(2\alpha-1)\beta+1}z^n \text{ for some } n.$$

This is due to Mogra, Reddy and Juneja [4].

Corollary 2.3: If $f\in\sigma_p(\alpha,1,1)$ then f is meromorphically convex of order $\delta(0\le\delta<1)$ in $|z|<r=r(\alpha,\delta)$

$$=\inf_n\left[\frac{(n+\alpha)(1-\delta)}{n(n+2-\delta)(1-\alpha)}\right]^{1/n+1}, n=1,2,....$$

The result is sharp for

$$f_n(z)=\frac{1}{z}+\frac{1-\alpha}{n+\alpha}z^n \text{ for some n.}$$

This is due to Juneja and Reddy [5].

5. Convex Linear Combinations

In this section we shall prove that the class $\sigma_p(\alpha,\beta,\xi)$ is closed under convex linear combinations.

Theorem 2.5: Let $f_0(z)=\frac{1}{z}$ and

$$f_n(z)=\frac{1}{z}+\frac{2\beta\xi(1-\alpha)}{(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1}z^n \quad \text{n}=1,2,.....$$

Then $f(z)\in\sigma_p(\alpha,\beta,\xi)$ if and only if, it can be expressed in the form

$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z)$ where $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty}\lambda_n=1$.

Proof: Let $f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z)$ with $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty}\lambda_n=1$.

Then

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z)=\lambda_0 f_0(z)+\sum_{n=1}^{\infty}\lambda_n f_n(z)$$

$$=\left[1-\sum_{n=1}^{\infty}\lambda_n\right]f_0(z)+\sum_{n=1}^{\infty}\lambda_n f_n(z)$$

$$=\left[1-\sum_{n=1}^{\infty}\lambda_n\right]\frac{1}{z}+\sum_{n=1}^{\infty}\lambda_n\left[\frac{1}{z}+\frac{2\beta\xi(1-\alpha)}{(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1}z^n\right]$$

$$=\frac{1}{z}+\sum_{n=1}^{\infty}\lambda_n\frac{2\beta\xi(1-\alpha)}{(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1}z^n$$

since

$$\sum_{n=1}^{\infty}\frac{(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1}{2\beta\xi(1-\alpha)}\lambda_n\frac{2\beta\xi(1-\alpha)}{(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1}$$

$$=\sum_{n=1}^{\infty}\lambda_n=1-\lambda_0\leq 1.$$

Therefore $f(z)\in\sigma_p(\alpha,\beta,\xi)$.

Conversely, suppose $f(z)\in\sigma_p(\alpha,\beta,\xi)$. Since

$$a_n\leq\frac{2\beta\xi(1-\alpha)}{(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1},\quad n=1,2,......$$

Setting

$\lambda_n=\frac{(1-\beta+2\beta\xi)n+(2\alpha\xi-1)\beta+1}{2\beta\xi(1-\alpha)}a_n$, n = 1, 2, …… and $\lambda_0=1-\sum_{n=0}^{\infty}\lambda_n$.

it follows that $f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z)$.

This completes the proof of the theorem.

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