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T-Pure Fuzzy Submodules

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Abstract: ***The main aim of this paper is to extend and study the notion of (ordinary) T-pure submodule into T-pure fuzzy submodule and T-pure ideal into T-pure fuzzy ideal. This lead us to introduced and study other notions such as T-pure fuzzy submodule and T-pure fuzzy ideal***

Keywords: Fuzzy module, submodules

1. Introduction

Let X be a fuzzy module of an R-module M, we denoted by X-F(M), it is well known that A is fuzzy submodules of X denoted by F-S(X) is called T-pure fuzzy submodule of X. if for each fuzzy ideal **I** of R such that $\boldsymbol{I^2}\mathrm{X} \cap \boldsymbol{A} = \boldsymbol{I^2 A}$. And an fuzzy ideal I of a ring R denoted by F- I(R) is called T-pure fuzzy ideal of R if for each fuzzy ideal $\boldsymbol{J^2}$of R $\boldsymbol{J^2} \cap \boldsymbol{I}$=$\boldsymbol{J^2 I}$

In this paper, we fuzzify these concepts T-pure fuzzy submodule and T-pure fuzzy ideal , moreover we generalize many properties of T-pure fuzzy submodule and T-pure fuzzy ideal

This paper consists of two part. In part one, various basis properties about T-pure fuzzy submodule are discussed . part two included T-pure fuzzy ideal and basic properties about this concept

1.1 T-pure Fuzzy submodule

In this section we inerdue the concept of T-pure fuzzy submodule by of provided some properties of this concepts.

Definition (1.1): Let X-F(M) .let A be a F-S(X). A is called a T-pure fuzzy submodule if for each fuzzy ideal **K** of R, $\boldsymbol{KX} \cap \boldsymbol{A} = \boldsymbol{KA}$. [4]

Proposition (1.2):
Let X-F(M) ,and let A be fuzzy sub modules of X. Then A is a T-pure F-S(X) $\Leftrightarrow$ $\boldsymbol{A_t}$ is a T-pure submodules of $\mathbf{X_t}$. $\forall$ t $\in$ (0,1]

Definition (1.3):
Let X-F(M) and let A be a F-S(X). A is called T-pure F-S(X) . if for each fuzzy ideal **I** of R such that $\boldsymbol{I^2}\mathrm{X} \cap \boldsymbol{A} = \boldsymbol{I^2 A}$.

Proposition (1.4):
Let X-F(M) and let A be fuzzy sub modules of X. Then A is T-pure fuzzy submodule of X if and only if $\boldsymbol{A_t}$ is T-pure submodules of $\mathbf{X_t}$. $\forall$ t $\in$ (0,1]

Proof:
Let J be an ideal of ring R

Define $\boldsymbol{I^2}: \boldsymbol{R} \rightarrow [\mathbf{0},\mathbf{1}]$ $\boldsymbol{by}$ $\boldsymbol{I^2(x)} = \begin{cases} \boldsymbol{t} & \boldsymbol{if\ x \in J} \\ \boldsymbol{o} & \boldsymbol{otherwise} \end{cases}$ $\forall$ t∈(0,1]

And let N ≪ M

Define A: M⟶[0,1] by $\boldsymbol{A(x)} = \begin{cases} \boldsymbol{t} & \boldsymbol{if\ x \in N} \\ \boldsymbol{o} & \boldsymbol{otherwise} \end{cases}$ $\forall$ t∈(0,1]

It is clear that $\boldsymbol{I^2}$ is F-I(R) and A is F-S(X).

Now, $\boldsymbol{A_t = N}$, $\boldsymbol{I_t^2 = J}$, $\boldsymbol{X_t = M}$

($\Longrightarrow$) Let A is T-pure fuzzy submodule of X .To prove A_t is T-pure submodules of X_t.
$\forall$ t∈(0,1].

To show that $I_t^2 X_t \cap A_t = A_t I_t^2$

$I_t^2 X_t \cap A_t$= $(I^2X)_t \cap A_t$ by[6]
= $(I^2 X \cap A)_t$ by[1]
=$(I^2 A)_t$ since A is T-pure
=$A_t I_t^2$

Thus $\boldsymbol{A_t}$ is T-pure submodules of $\boldsymbol{X_t}$. $\forall$ t∈(0,1].

Conversely Let I^2 be F-I(R) and A be a F-S(X).

T.p A is T-pure fuzzy submodule of X

$(\boldsymbol{I^2X} \cap \boldsymbol{A})_t$=$(\boldsymbol{I^2X})_t \cap \boldsymbol{A_t}$ $\forall$ t∈(0,1].
=$\boldsymbol{I_t^2 X_t} \cap \boldsymbol{A_t}$

but $\boldsymbol{A_t}$ is T-pure submodules of $\boldsymbol{X_t}$.

Then $\boldsymbol{I_t^2 X_t} \cap \boldsymbol{A_t}$=$\boldsymbol{I_t^2 A_t}$
= $(\boldsymbol{I^2 A})_t$

Hence$(\boldsymbol{I^2X} \cap \boldsymbol{A})_t$=$(\boldsymbol{I^2A})_t$

$\boldsymbol{I^2X} \cap \boldsymbol{A}$ =$\boldsymbol{I^2A}$

Therefore A is T-pure fuzzy submodule of X .

Remarks and Examples (1.5):

1- Let X-F(M) and let A be a pure F-S(X) , Then A is T-pure F-S(X).

<u>Proof</u>:
It is clear

The converse not true by

<u>Example:</u> Let M=$\boldsymbol{Z_4}$ as Z-module and N=2$\boldsymbol{Z_4}$

Define X: M⟶[0,1] by $\boldsymbol{X(x)} = \begin{cases} \boldsymbol{1\ if\ x \in M} \\ \boldsymbol{o\ otherwise} \end{cases}$

Define A: M⟶[0,1] by $\boldsymbol{A(x)} = \begin{cases} \boldsymbol{t\ if\ x \in N} \\ \boldsymbol{o\ otherwise} \end{cases}$ $\forall$ t∈(0,1]

It is clear that $\boldsymbol{X}$ is F(M) , A is F-S(X) and $\boldsymbol{X_t}$=M ,$\boldsymbol{A_t}$=N

$\boldsymbol{A_t}$ is T-pure submodules of $\boldsymbol{X_t}$. by[7]

Thus A is T-pure fuzzy submodule of X by (Proposition1.4)

But A is not pure fuzzy submodule of X since if $\boldsymbol{I_t}$=2Z where $\boldsymbol{I}$:R⟶[0,1]

Such that $\boldsymbol{I}$(x)=t if x∈2Z and $\boldsymbol{I}$(x)=0 if x∉2Z

Now 2Z.4Z∩ $\mathbf{2}\boldsymbol{Z_4}$={$\overline{\mathbf{0}}, \overline{\mathbf{2}}$} but 2Z.$\mathbf{2}\boldsymbol{Z_4}$=2{$\overline{\mathbf{0}}, \overline{\mathbf{2}}$}={$\overline{\mathbf{0}}$}

Thus $\boldsymbol{A_t}$ is not pure submodules of $\boldsymbol{X_t}$

Therefore A is not pure fuzzy submodule of X. [4]

2- Let X-F(M) . It is clear that the fuzzy singleton $\{\boldsymbol{o_t}\}$ and X are always T-pure fuzzy submodule of X. ∀ t∈(0,1]

3-In the fuzzy module Z as Z-module. The only T-pure fuzzy submodule are fuzzy singleton $\{\boldsymbol{o_t}\}$ and X

Proof:

Let X: Z⟶[0,1] by $\boldsymbol{X}(\boldsymbol{x}) = \begin{cases} \boldsymbol{1\ if\ x \in Z} \\ \boldsymbol{o\ otherwise} \end{cases}$

Define $\boldsymbol{O_t}$: X⟶[0,1] by $\boldsymbol{O_t}(\boldsymbol{x}) = \begin{cases} \boldsymbol{t\ if\ x = 0} \\ \boldsymbol{o\ otherwise} \end{cases}$ ∀ t∈(0,1]

$\boldsymbol{X_t}$=Z and $\boldsymbol{O_t}$=0

By(2) clear that $\boldsymbol{X}$ and $\boldsymbol{O_t}$are T-pure fuzzy submodule

If there exists a fuzzy submodule

A:nZ⟶[0,1] by $\boldsymbol{A}(\boldsymbol{x}) = \begin{cases} \boldsymbol{t\ if\ x \in nZ} \\ \boldsymbol{o\ otherwise} \end{cases}$ ∀ t∈(0,1]

$\boldsymbol{A_t}$=nZ

Let $\boldsymbol{I}$(x):$\langle \boldsymbol{n} \rangle^2$ ⟶[0,1] by $\boldsymbol{I}(\boldsymbol{x}) = \begin{cases} \boldsymbol{t\ if\ x \in \langle n \rangle^2} \\ \boldsymbol{o\ otherwise} \end{cases}$ ∀ t∈(0,1]

It is clear that $\boldsymbol{I_t}$=$\langle \boldsymbol{n} \rangle^2$ and I is a fuzzy ideal

$\boldsymbol{n^2}$=$\boldsymbol{n^2}.\boldsymbol{1\epsilon}\langle \boldsymbol{n} \rangle^2$Z∩nZ= $\langle \boldsymbol{n} \rangle^2$.nZ=$\boldsymbol{n^3}$Z

But $\boldsymbol{n^2} \notin \boldsymbol{n^3}$Z Thus $\boldsymbol{A_t}$ is not T- pure by [7]

$\boldsymbol{X_t}$=Z , $\boldsymbol{O_t}$=0 only two T-pure submodule of Z-module ∀ t∈(0,1] by (Proposition1.4)

4-Let X be a fuzzy module of an Z-module Q. and let A be a non-empty FC-S(X) . then A is not T-pure F-S(X).

Proof:

Define X: Q⟶[0,1] by $\boldsymbol{X}(\boldsymbol{x}) = \begin{cases} \boldsymbol{1\ if\ x \in Q} \\ \boldsymbol{o\ otherwise} \end{cases}$

Define A: Q⟶[0,1] by $\boldsymbol{A}(\boldsymbol{x}) = \begin{cases} \boldsymbol{1\ if\ x \in N} \\ \boldsymbol{o\ otherwise} \end{cases}$ ∀ t∈(0,1]

where $\boldsymbol{N}$ is submodule of Q, $\boldsymbol{X_t}$=Q and $\boldsymbol{A_t}$=N

N is not T- pure fuzzy submodule of Q by [7]

Then A is not T-pure F-S(X) by (Proposition(1.4))

5- Let X-F(M) . let A be a T-pure F-S(X) such that A≅B where $\boldsymbol{B}$ is F-S(X) , then B is not T- pure F-S(X) for example.

Example: Let M=Z

Let X: M⟶[0,1] by $\boldsymbol{X}(\boldsymbol{x}) = \begin{cases} \boldsymbol{1} & \boldsymbol{if\ x \in M} \\ \boldsymbol{o} & \boldsymbol{otherwise} \end{cases}$

Let A: Z⟶[0,1] by $\boldsymbol{A}(\boldsymbol{x}) = \begin{cases} \boldsymbol{t} & \boldsymbol{if\ x \in Z} \\ \boldsymbol{o} & \boldsymbol{otherwise} \end{cases}$ ∀ t∈(0,1]

Let B: Z⟶[0,1] by $\boldsymbol{B}(\boldsymbol{x}) = \begin{cases} \boldsymbol{t} & \boldsymbol{if\ x \in 2Z} \\ \boldsymbol{o} & \boldsymbol{otherwise} \end{cases}$ ∀ t∈(0,1]

It is clear that A and B are F-S(X), Now $\boldsymbol{A_t}$=Z and $\boldsymbol{B_t}$=2Z and Z≅2Z but 2Z is not T- pure submodules [7]

B is not T-pure fuzzy submodule by (Proposition(1.4))

Proposition (1.6):

Let X-F(M), and let A and B are two F-S(X) . if A is T-pure F-S(X), B ⊆ $\boldsymbol{A}$. and B is T-pure F-S(A) , then B is T-pure fuzzy submodule of X.

Proof:

Since A be a T-pure F-S(X) then $\boldsymbol{I^2}$X ∩ $\boldsymbol{A} = \boldsymbol{I^2 A}$…..(1)

where $\boldsymbol{I^2}$ is F-I(R) and since B be a T-pure F-S(A) then

$\boldsymbol{I^2}$A ∩ $\boldsymbol{B} = \boldsymbol{I^2 B}$….(2)

Now, we get $\boldsymbol{I^2 B} =$ $\boldsymbol{I^2}$A ∩ $\boldsymbol{B}$….(2)

= ($\boldsymbol{I^2}$X ∩ $\boldsymbol{A}$)∩B…..(1)

= $\boldsymbol{I^2}$X ∩ ($\boldsymbol{A}$ ∩B) since B ⊆ A

Therefore B is T-pure fuzzy submodule of X .

Proposition(1.7):

Let X-F(M) . and let C be a T-pure F-S(X). If B is a F-S(X) contating A, then A is T-pure F-S(B)

Proof :

Let $\boldsymbol{I^2}$ be a F-I(R) and let C be a T-pure F-S(X)

Hence $\boldsymbol{I^2}$X ∩ $\boldsymbol{C} = \boldsymbol{I^2 C}$

Now, $\boldsymbol{I^2}$B ∩ $\boldsymbol{C}$ = ($\boldsymbol{I^2}$B ∩ $\boldsymbol{I^2 X}$) ∩ $\boldsymbol{C}$ since $\boldsymbol{C}$ ⊆B⊆X

=$\boldsymbol{I^2}$B ∩ ($\boldsymbol{I^2 X}$ ∩ $\boldsymbol{C}$)

=$\boldsymbol{I^2}$B ∩ $\boldsymbol{I^2 C}$

=$\boldsymbol{I^2 C}$

Thus C is T-pure fuzzy submodule of a fuzzy submodule B .

"Definition (1.8):

Let X ,Y -F(M_1, M_2)respectively. Define X⊕Y: M_1⊕M_2 ⟶ [0,1] by $(X⊕Y)(a, b) = \min\{X(a), Y(b)$ for all $(a, b) \in M_1⊕M_2\}$ X⊕Y is called a fuzzy external direct sum of X and Y. [9]"

Lemma(1.9):

Let $\boldsymbol{N_1}$ and $\boldsymbol{N_2}$ be two S($\boldsymbol{M_1}$) and S ($\boldsymbol{M_2}$) if $\boldsymbol{N_1} \oplus \boldsymbol{N_2}$ is T-pure submodule of $\boldsymbol{M_1} \oplus \boldsymbol{M_2}$ then $\boldsymbol{N_1}$ and $\boldsymbol{N_2}$ are T-pure submodule in $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$.

Proof:

T.p $\boldsymbol{I^2}$ $\boldsymbol{M_1} \cap \boldsymbol{N_1} = \boldsymbol{I^2 N_1}$ and $\boldsymbol{I^2}$ $\boldsymbol{M_2} \cap \boldsymbol{N_2} = \boldsymbol{I^2 N_2}$ for each ideal $\boldsymbol{I^2}$ of R.

Since $\boldsymbol{N_1} \oplus \boldsymbol{N_2}$ is T-pure in $\boldsymbol{M_1} \oplus \boldsymbol{M_2}$ we get:

$\boldsymbol{I^2}(\boldsymbol{M_1} \oplus \boldsymbol{M_2}) \cap (\boldsymbol{N_1} \oplus \boldsymbol{N_2}) = \boldsymbol{I^2}(\boldsymbol{N_1} \oplus \boldsymbol{N_2})$

$(\boldsymbol{I^2}\ \boldsymbol{M_1} \oplus \boldsymbol{I^2 M_2}) \cap (\boldsymbol{N_1} \oplus \boldsymbol{N_2}) = \boldsymbol{I^2 N_1} \oplus \boldsymbol{I^2 N_2}$

Hence $\boldsymbol{I^2}$ $\boldsymbol{M_1} \cap \boldsymbol{N_1} = \boldsymbol{I^2 N_1}$ and $\boldsymbol{I^2 M_2} \cap \boldsymbol{N_2}$=$\boldsymbol{I^2 N_2}$

Thus $\boldsymbol{N_1}$ and $\boldsymbol{N_2}$ are T-pure.

Proposition(1.10):

Let X_1-F(M_1) and X_2-F(M_2), If A , B be are two F-S(X_1) and F-S(X_2). respectiveely then A and B are T-pure fuzzy submodule of X_1 $and X_2$ if and only if A⊕B is T-pure fuzzy submodule of $X_1 \oplus X_2$.

Proof:

(⟹) $\boldsymbol{A_t}$⊕$\boldsymbol{B_t}$=(A⊕$\boldsymbol{B}$) $_t$ and ($\boldsymbol{X_1}_t$⊕$\boldsymbol{X_2}_t$)=($\mathbf{X_1} \oplus \mathbf{X_2})_t$. ∀$\boldsymbol{t} \in (\boldsymbol{0}, \boldsymbol{1}]$ by [4]

Therefore (A⊕$\boldsymbol{B}$)$_t$ is T-pure of ($\boldsymbol{X_1} \oplus \boldsymbol{X_2})_t$ by [7]

Thus A⊕B is T-pure fuzzy submodule of $X_1 \oplus X_2$. (Proposition(1.4))

⟸ let A⊕$\boldsymbol{B}$ is T-pure fuzzy submodule of $\boldsymbol{X_1} \oplus \boldsymbol{X_2}$.

To show that A and B are T-pure fuzzy submodule of $\boldsymbol{X_1}, \boldsymbol{X_2}$ respectively.

By [4, lemma (2.2.4)] and (Proposition1.4) we get :-

(A⊕$\boldsymbol{B}$)$_t$=$\boldsymbol{A_t}$⊕$\boldsymbol{B_t}$ is T-pure in module $(\boldsymbol{X_1})_t$⊕$(\boldsymbol{X_2})_t$

Thus $\boldsymbol{A_t}$ and $\boldsymbol{B_t}$are T-pure submodule of $(\boldsymbol{X_1})_t$ **and** $(\boldsymbol{X_2})_t$ by (lemma (1.9))

Therefore A and B are two T-pure fuzzy submodules of a fuzzy modules $\boldsymbol{X_1}$ ***and*** $\boldsymbol{X_2}$ by (Proposition(1.4)).

Proposition(1.11):

Let H be a direct summand of a fuzzy module X. then H is T-pure fuzzy submodule of X.

Proof:
Let X $=H \oplus C$, where C is a F-S(X) and H is a direct summand of X.
Such that X=H+C and H $\cap C = 0$ by[def of fuzzy direct summand]
To prove H is T-pure (i.e I^2X ∩ H=I^2H for each I^2 is F-I(R))
I^2X ∩ H=$I^2(H \oplus C) \cap H$
=$(I^2H \oplus I^2C) \cap (H \oplus 0)$
=$(I^2H \cap H) \oplus (I^2C \cap 0)$ by[4]
=$(I^2H \cap H) \oplus 0$
= $I^2H \cap H$
=I^2H since $I^2H \subseteq H$
Therefore H is T-pure fuzzy submodule of X.

Proposition(1.12):
let f :X $\longrightarrow$ **Y** be epimorphism of X_1-F(M_1) and X_2-F(M_2) respectively, let B be a fuzzy submodule of X and X is f-invariant , if B is a T-pure F-S(X) ,then f(B) is T-pure F-S(Y).

Proof:
To prove $I^2Y \cap f(B) = I^2 f(B)$ for each fuzzy ideal I of R.
$I^2Y \cap f(B)$=$I^2 f(X) \cap f(B)$ since f is epimorphism
=$f(I^2X) \cap f(B)$ by[4]
= $f(I^2X \cap B)$ by[3]
= $f(I^2B)$ since B is T-pure
=$I^2 f(B)$ by[4]
Thus f(B) is T-pure fuzzy submodule of Y.

Proposition(1.13):
let f :X $\longrightarrow$ **Y** be epimorphism of X_1-F(M_1)and X_2-F(M_2) respectively, such that every submodule of X is f-invariant , if C is T-pure F-S(Y), then f^{-1} (C) is T-pure fuzzy submodule of X.

Proof:
To prove f^{-1} (C) is T-pure (i.e I^2X ∩ $f^{-1}(C)$ =I^2 f^{-1}(C) for each I^2 is F-I(R)).
f (I^2X ∩ f^{-1}(C)) = f(I^2X) ∩ f(f^{-1}(C)) by[3]
=$I^2 f(X) \cap C$ by[4]
=$I^2Y \cap C$ since f is epimorphism
= I^2C since A is T-pure
Therefore f (I^2X ∩ f^{-1}(C)) = I^2C so f^{-1}[f (I^2X ∩ f-1C)= f-1(I2C)
But f^{-1}[f(I^2X ∩ f^{-1}(A))] = I^2X ∩ f^{-1}(C) since by hyposse and
f^{-1}(I^2C)= I^2 f^{-1}(C) by[4]
Thus I^2X ∩ f^{-1}(C) =I^2 f^{-1}(C)

Lemma(1.14):
Let$\{I_i, i \in N\}$ be an ascending chain of T-pure fuzzy submodules of a F(X) and let A be a F-I(R), then $A^2[\cup_{i\in N} I_i] = \cup_{i\in c}[A^2 I_i]$.

Proof:
Since $A^2 I_i \subseteq A^2[\cup_{i\in N} I_i], \forall\, i \in N$
implies that $\cup_{i\in c}[A^2 I_i] \subseteq A^2[\cup_{i\in N} I_i]$……(1)
and by $A^2[\cup_{i\in N} I_i] \subseteq A^2$ and $A^2[\cup_{i\in N} I_i] \subseteq \cup_{i\in N} I_i$
since $\cup_{i\in N} I_i$ is fuzzy submodule of X
then $A^2[\cup_{i\in N} I_i] \subseteq A^2X \cap [\cup_{i\in N} I_i]$
=$\cup_{i\in c}$ [$A^2X \cap I_i$] by[4]
=$\cup_{i\in c}$ [A^2I_i] since I_i T-pure fuzzy submodule
Thus $A^2[\cup_{i\in N} I_i] \subseteq \cup_{i\in c}$ [A^2I_i]
Therefore $A^2[\cup_{i\in N} I_i] = \cup_{i\in c}[A^2 I_i]$

Proposition(1.15):
If $\{I_i, i \in N\}$ be an ascending chain of T-pure fuzzy submodule of fuzzy module X , then $\cup_{i\in N} I_i$ is T-pure fuzzy submodule of X .

Proof:
We must prove that $\cup_{i\in N} I_i$ is T-pure i.e $C^2X \cap [\cup_{i\in N} I_i] = C^2[\cup_{i\in N} I_i]$ for each fuzzy ideal C^2 of R
$C^2X \cap [\cup_{i\in N} I_i]$ =$\cup_{i\in c}$ [$C^2X \cap I_i$] by [4]
=$\cup_{i\in c}$ [C^2I_i] since I_i T-pure F-I(R)
= $C^2[\cup_{i\in N} I_i]$ by(lemma1.14)
Thus $\cup_{i\in N} I_i$ is T-pure submodule of X .

2. T-pure fuzzy ideal

Definition (2.1):
An fuzzy ideal I of a ring R is called T-pure F-I(R) if for each fuzzy ideal $J^2$of R $J^2 \cap I$= J^2I

Definition (2.2):
If every F-I(R) is T-pure fuzzy ideal ,then we say R is T-regular fuzzy ring.

Proposition (2.3):
Let I be a fuzzy ideal of R then I is T-pure if and only if I_t is a T-pure ideal of R. ∀ t∈(0,1]

Proof:
(⟹) Let I is T-pure F-I(R) T.p I_t is a T-pure ideal of R ∀ t∈(0,1]
Let J^2 be an ideal of R
Define K^2: J^2 ⟶[0,1] by $K^2(x) = \begin{cases} t & if\ x \in J^2 \\ o & otherwise \end{cases}$ ∀ t∈(0,1]
It is clear that K^2 is F-I(R) and $K^2_t = J^2$
T.P $J^2 \cap I_t$=J^2I_t
$J^2 \cap I_t$=$K^2_t \cap I_t$
=$(K^2 \cap I)_t$
=$(K^2 I)_t$ since I is T-pure ideal
=$K^2_t I_t$
=$J^2 I_t$ ∀ t∈(0,1]
⟸ I_t is a T-pure ideal T.p I is T-pure fuzzy ideal
Let K^2 is fuzzy ideal of R T.p $K^2 \cap I$=K^2I
$(K^2 \cap I)_t$=$K^2_t \cap I_t$
=$K^2_t I_t$
=$(K^2 I)_t$ sinceI_t is T-pure ideal
Thus $(K^2 \cap I)_t = (K^2 I)_t$
Hence $K^2 \cap I$=K^2I
I is T-pure fuzzy ideal of R.

Remarks and Examples(2.4):
1-Let X-F(M), . R is regular fuzzy ring then R is T-regular fuzzy ring.

Proof:
It is clear that
The converse not true by

Example: Let M=$\boldsymbol{Z_4}$ as Z-module and N={$\bar{0}, \bar{2}$}

Define X: M⟶[0,1] by $X(x) = \begin{cases} 1 \text{ if } x \in M \\ o \text{ otherwise} \end{cases}$

Define I: R⟶[0,1] by $A(x) = \begin{cases} t \text{ if } x \in N \\ o \text{ otherwise} \end{cases}$ ∀ t∈(0,1]

It is clear that I is fuzzy ideal of R and X_t=M ,I_t=N

Then X_t is T-regular ring by [7]

Hence X is T- regular fuzzy ring since every ideal of Z_4 is T-pure by (Proposition1.4) every ideal is fuzzy T-pure.

But X_t is not regular ring since ideal {$\bar{0}, \bar{2}$} is not pure [7]

Thus I not pure fuzzy ideal by [4]

Then X is not regular fuzzy ring .

2- Let X-F(M), if R is a ring then the fuzzy singleton {o_t} and a ring R are always T-pure F-I(R)

3- Let X-F(M) . if R is a field , then X is T- regular ring.

Proof:

Since every field has only one submodule 0 then by(2), X is T- regular ring.

The converse of (3) is true if we gives the condition, R is a fuzzy integral domain

Proposition (2.5):

Let R_1 , R_2 be two rings and Let g any epimorphism function from R_1 to R_2. If C be a T-pure F-I(R_1) , then g(C) is a T-pure F-I(R_2).

Proof:

Let I^2 be a F-I(R_2). To prove g(C) ∩ I^2=I^2g(C).

g(C) ∩ I^2= g(C) ∩ g^{-1}(g (I^2)). [5]

= g(C ∩ g(I^2)) [3]

But g^{-1}(I^2) is F-I(R_1) by [5]

And C is T-pure F-I(R_1) so that

g(C ∩ g^{-1}(I^2))= g(A. g^{-1}(I^2))

= g(C). g (g^{-1}(I^2)) by [2]

= g(C). I^2 by[5]

Proposition (2.6):

Let R_1 , R_2 be two rings and Let f any epimorphism function from R_1 to R_2 and every fuzzy ideal of R_1 is f-invariant ,then if C is T-pure fuzzy ideal of R_2. Then $f^{-1}(C)$ is T-pure F-I(R_1) .

Proof:

Let C be a F-I(R_2). then $f^{-1}(C)$ is F- I(R_1) .see [5]

Let J^2 be a F-I(R_2). To. Prove $f^{-1}(C) \cap J^2 = f^{-1}(C) J^2$

And by $f(f^{-1}(C) \cap J^2) = f(f^{-1}(C) \cap f(J^2)$ see[3]

=C ∩ $f(J^2)$ see[5]

=C. $f(J^2)$ since C is T-pure

= $f(f^{-1}(C) f(J^2)$ by [5]

= $f(f^{-1}(C) J^2)$ by [2]

Hence

f^{-1} [$f(f^{-1}(C) J^2)$] =f^{-1} [$f(f^{-1}(C) \cap J^2)$]

and by hypposе. We get

$f^{-1}(C) \cap J^2 = (f^{-1}(C) J^2$

Therefore $f^{-1}(C)$ is T-pure F-I(R_1).

Proposition (2.7):

Let K be a is F-I(R_1) and let J be a F-I(R_2), then K⊕J is T-pure fuzzy ideal of $R_1 \oplus R_2$ if and only if K and J are T-pure fuzzy ideal in R_1 *and* R_2 respectively.

Proof:

(⟹) Let K ⊕J is T-pure fuzzy ideal To. Prove K and J are T-pure fuzzy ideal

Let A^2 and B^2 be two fuzzy ideal of R_1 *and* R_2 respectively. Then A^2⊕ B^2 is T-pure fuzzy ideal of $R_1 \oplus R_2$ see[4]

Hence (K ⊕J) ∩ (A^2⊕ B^2)= (K ⊕J) (A^2⊕ B^2) since (K ⊕J) is T-pure fuzzy ideal

And by (K ⊕J) ∩ (A^2⊕ B^2)= (K ∩ A^2) ⊕(J ∩ B^2) see[4]

(K ⊕J) (A^2⊕ B^2)= (KA^2) ⊕(J B^2) see[4]

There for (K ∩ A^2) = KA^2 and J ∩ B^2=J B^2 see[4]

Thus K and J are T-pure fuzzy ideals of R_1 *and* R_2.

(⟸) let K and J are T-pure fuzzy ideal of R_1 *and* R_2.

Let A^2 and B^2 be two fuzzy ideal of R_1 *and* R_2

Hence A^2⊕ B^2 is fuzzy ideal in $R_1 \oplus R_2$ see[8]

T.p (K ⊕J) ∩ (A^2⊕ B^2)= (K ⊕J) (A^2⊕ B^2)

(K ⊕J) ∩ (A^2⊕ B^2)= (K ∩ A^2) ⊕(J ∩ B^2) see [4]

=(KA^2) ⊕(J B^2) since K *and* J are T-pure

=(K ⊕J) (A^2⊕ B^2) see[4]

Hence (K ⊕J) ∩ (A^2⊕ B^2)= (K ⊕J) (A^2⊕ B^2)

Thus K ⊕J is T-pure fuzzy ideal in $R_1 \oplus R_2$

Proposition2.8:

Let I and J are T-pure F-I(R) , then I ∩ J is T-pure F-I(R).

Proof:

T.p I ∩ J is T-pure F-I(R).

To show that for each fuzzy ideal K^2 of R (I ∩ J) ∩ K^2= (I ∩ J) K^2

Now, ∀ t ∈ (0,1] $((I \cap J)K^2)_t = (I \cap J)_t K_t^2$ by[10]

=$(I_t \cap J_t)K_t^2$ by[1]

=$(I_t J_t)K_t^2$ since J_t is T-pure (by level)

=$I_t (J_t K_t^2)$

=$I_t (JK^2)_t$ by [10]

=$I_t (J \cap K^2)_t$ since J is T-pure

= $I_t \cap (J \cap K^2)_t$ since I_t is T-pure

=$I_t \cap (J_t \cap K_t^2)$ by[1]

=$(I_t \cap J_t) \cap K_t^2$

= $((I \cap J)_t \cap K_t^2)$ by [1]

=$[(I \cap J) \cap K^2]_t$ by [1]

Therefore (I ∩ J) ∩ K^2=(I ∩ J)K^2

Thus I ∩ J is T-pure fuzzy ideal of R

Lemma2.9:

let {$J_i, i \in N$} be an ascending chain of T-pure F-I(R) . let C be a fuzzy ideal of R,

then $C^2[\cup_{i \in N} J_i] = \cup_{i \in c}[C^2 J_i]$.

proof:

$C^2 J_i \subseteq C^2[\cup_{i \in N} J_i], \forall i \in N$

implies that $\cup_{i \in c}[C^2 J_i] \subseteq C^2[\cup_{i \in N} J_i]$……(1)

but $C^2[\cup_{i \in N} J_i] \subseteq C^2$ and $C^2[\cup_{i \in N} J_i] \subseteq \cup_{i \in N} J_i$

since $\cup_{i \in N} J_i$ is fuzzy ideal

on other side $C^2[\cup_{i \in N} J_i] \subseteq C^2 \cap [\cup_{i \in N} J_i]$

=$\cup_{i \in c}$ [$C^2 \cap J_i$] by [4]

=$\cup_{i \in c}$ [$C^2 J_i$] since J_i T-pure fuzzy ideal of R

Thus $C^2[\cup_{i \in N} J_i] \subseteq \cup_{i \in c}$ [$C^2 J_i$]

Therefore $C^2[\cup_{i \in N} J_i] = \cup_{i \in c}[C^2 J_i]$

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Proposition2.10:
If $\{\boldsymbol{J_i}, \boldsymbol{i} \in \boldsymbol{N}\}$ be an ascending chain of T-pure F-I($\boldsymbol{R}$), then $\cup_{i\in N} \boldsymbol{J_i}$ is T-pure F-I($\boldsymbol{R}$).

Proof:
We must show that for each fuzzy ideal $\boldsymbol{C^2}$ of R

$\mathbf{C^2} \cap [\cup_{i\in N} \mathbf{J_i}] = \mathbf{C^2}[\cup_{i\in N} \mathbf{J_i}]$

$\mathbf{C^2} \cap [\cup_{i\in N} \mathbf{J_i}] = \cup_{i\in c} [\boldsymbol{C^2} \cap \boldsymbol{J_i}]$ by [4]

$= \cup_{i\in c} [\boldsymbol{C^2}\boldsymbol{J_i}]$ since $\boldsymbol{J_i}$ T-pure F-I($\boldsymbol{R}$).

$= \boldsymbol{C^2}[\cup_{i\in N} \boldsymbol{J_i}]$ by lemma(2.9)

Hence $\cup_{i\in N} \boldsymbol{J_i}$ is T-pure F-I($\boldsymbol{R}$).

References

[1] Zadeh L. A., ''Fuzzy Sets, Information and control'' ,Vol. 8,PP.338-353,(1965).
[2] Zahedi M .M., ''A characterization of L-Fuzzy Prime Ideals '' , Fuzzy Sets and Systems, Vol. 44 ,147-160,(1991).
[3] Zhao Jiandi, Shi K . Yue M., ''Fuzzy Modules over Fuzzy Rings'', The J. of Fuzzy Math. Vol. 3,PP. 531-540,(1993).
[4] Maysoun A. H., ''F-Regular Fuzzy Modules'', M. Sc. Thesis , University of Baghdad,(2002).
[5] Kumar R., ''Fuzzy Semi primary Ideal of Rings''. Fuzzy Sets and Systems, Vol. 42, PP. 263-272,(1991).
[6] Kumar R., ''Fuzzy Cosets Some Fuzzy Radicals'', Fuzzy Sets and Systems, vol. 46,PP.261-265,(1992).
[7] Ghaleb A. H., ''Generalization of Regular Module and Pure Submodules''. Ph.D. Thesis , University of Baghdad,(2015).
[8] Areeg T. H., ''On Almost Quasi- Frobenius Fussy Rings'', M. Sc. Thesis , University of Baghdad,(2000).
[9] Rabi H.J. ,''Prime Fuzzy Submodule and Prime Fuzzy Modules'' ,M. Sc. Thesis, University of Baghdad,(2001).
[10] Inaam M-A.H, ''On Fuzzy Ideals of Fuzzy Rings'', Math. And Physics, Vol. 16, PP.4, (2001)