International Journal of Science and Research (IJSR)
ISSN (Online): 2319-7064
Index Copernicus Value (2017): 78.96 | Impact Factor (2017): 6.391

Efficient Iterative Method for Initial and Boundary Value Problems Appear in Engineering and Applied Sciences

Majeed Ahmed AL-Jawary[1], Mustafa Mahmood Azeez[2]

[1,2]Department of mathematics, Baghdad University, College of Education for pure Sciences/ Ibn-AL-Haitham, Baghdad, Iraq

Abstract: ***The main aim and contribution of the current paper is to implement a semi-analytical iterative method proposed by Temimi and Ansari namely (TAM) to solve the Riccati, pantograph and elastic beam deformation equations, which appeared in models of various problems in engineering and applied sciences. The exact solutions are obtained for Riccati, Pantograph equations and an approximate solution for beam equation. The convergence of the TAM is investigated for the three problems. In general, the accuracy of our result for beam equation is better than those of Homotopy perturbation method (HPM) and Variational Iteration Method (VIM). The software used for the terms calculation in iterative process was MATHEMATICA® 10.***

Keywords: Differential equations, Riccati equation, Pantograph equation, Elastic beam deformation equations, Iterative method

1. Introduction

When you submit your paper print it in two-column format, including figures and tables [1]. In addition, designate one author as the "corresponding author". This is the author to whom proofs of the paper will be sent. Proofs are sent to the corresponding author only [2].

The linear and nonlinear differential equations play an important role in many problems that occur in various areas of physics, chemistry, engineering and applied science. The past few decades have seen significant advances to implement analytic, approximate and numerical methods for solving linear and nonlinear differential equations, earlier studies [31]. Several methods have applied to solve linear and nonlinear ODEs and PDEs such as the Adomian decomposition method (ADM) [1], the variational iteration method (VIM) [37], homotopy perturbation method (HPM) and differential transform method (DTM) [17]. Although these methods achieve some useful solutions, however, some drawbacks have been appeared such as calculate Adomian polynomial to deal with nonlinear terms in ADM, calculating Lagrange multiplier in VIM in which the terms of the sequence became complicated after several iteration, construct a homotopy and solve the corresponding equations in HPM.

Riccati equation is an initial value problem of nonlinear differential equation which plays a significant role in many fields of applied science such as random processes, optimal control, diffusion problems, network synthesis and financial mathematics [7].

Also, pantograph equation is originated from the work of Ockendon and Tayler on the collection of current by the pantograph head of an electric locomotive [24]. The pantograph equations are appeared in modeling of various problems in engineering and sciences such as biology, economy, control and electrodynamics.

On the other hand, the beam deformation equation is a nonlinear boundary value problem (BVP) which is frequently used as mathematical model in viscoelastic, inelastic flows and deformation of beams [10, 38]. Recently, Temimi and Ansari have introduced a semi–analytical iterative method namely (TAM) for solving nonlinear problems [35]. The main feature of the TAM is does not any required restricted assumptions to deal with nonlinear terms, time saver and has a higher convergence and accuracy. The TAM was inspired from the homotopy analysis method (HAM) [28] and it is one of the famous iterative methods that used for solving nonlinear problems [12]. Moreover, this method has been successfully applied to solve other different problems [19, 4-6]. In this article, the application of TAM for solving the Riccati, pantograph and elastic beam deformation equations will be presented. The efficiency and accuracy has been proved by studying the convergence and error analysis.

Our work in this paper is organized as follows; in section two the basic idea of the TAM is presented. The convergence of the TAM is discussed in section three. The scientific applications with some examples are introduced and solved in section four. Finally the conclusion is given in section five.

2. The Basic Idea of TAM

We start by pointing out that nonlinear differential equation can be written as

We start by pointing out that nonlinear differential equation can be written as**:**

$$L(u(x))+N(u(x))+g(x) = 0\,, \qquad B(u,\tfrac{du}{dx}) = 0. \tag{1}$$

Where x denotes the independent variable, u(x) is an unknown function, g(x) is a known function, L is a linear operator, N is a nonlinear operator and B is a boundary operator. The main requirement here is that L is the linear part of the differential equation, but it is possible to take some linear parts and add them to N as needed. The method works in the following steps, starts by assuming that $u_0(x)$ is an initial guess of the solution to the problem [35].

$$L(u_0(x)) + g(x) = 0, \qquad B(u_0, \tfrac{du_0}{dx}) = 0. \qquad (2)$$

To generate the next iteration to the solution, we solve the following problem:

$$L(u_1(x)) + g(x) + N(u_0(x)) = 0, \qquad B(u_1, \tfrac{du_1}{dx}) = 0. \qquad (3)$$

Thus, we have a simple iterative procedure which is effectively the solution of a linear set of problems i.e.

$$L(u_{n+1}(x)) + g(x) + N(u_n(x)) = 0, \quad B(u_{n+1}, \tfrac{du_{n+1}}{dx}) = 0. \quad (4)$$

It is important to note that each $u_i(x)$ are solutions to the problem. That this procedure iterative although perfect to apply has merit in that each solution is an improvement of the previous iterate and as more iteration are taken the solution converges to the solution of problem.

3. The Convergence of the TAM

The Banach fixed-point theorem (also known as the contraction mapping theorem or contraction mapping principle) is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. The theorem is named after Stefan Banach (1892–1945), and was first stated by him in 1922 [9]. In this section, some basic concepts and the main theorem of the convergence will be presented.

Definition 3.1 Let (X, d) be a metric space. Then a map $T: X \to X$ is called a contraction mapping on X if there exists $k \in [0, 1)$ [9], such that

$$d(T(x), T(y)) \leq kd(x, y) \qquad \text{for all } x, y \text{ in } X.$$

Banach Fixed Point Theorem 3.2 Let (X, d) be a non-empty complete metric space with a contraction mapping $T: X \to X$. Then T admits a unique fixed-point x^* in X (i.e. $T(x^*) = x^*$). Furthermore, x^* can be found as follows: start with an arbitrary element x_0 in X and define a sequence $\{x_n\}$ by $x_n = T(x_{n-1})$, then $x_n \to x^*$ [9].

Theorem 3.3 Suppose that X and Y be Banach space and $N: X \to Y$. is a contraction nonlinear mapping, that is [13].

$$\forall\, v, v^* \in X;\ \|N(v) - N(v^*)\| \leq k\|v - v^*\|, \quad 0 < k < 1.$$

Which according to Banach's fixed point theorem, having the fixed point u, that is $N(u) = u$, the sequence generated by the TAM will be regarded as

$$u_n = N(u_{n-1}), \qquad u = \lim_{n\to\infty} u_n,$$

and suppose that $u_0 \in B_r(u)$ where $B_r(u) = \{u^* \in X, \|u^* - u\| < r\}$ then we have the following statements:

1. $\|u_n - u\| \leq k^n \|u_0 - u\|$,
2. $u_n \in B_r(u)$,
3. $\lim_{n\to\infty} u_n = u$,

Proof: See [13].

4. Application of the TAM with Convergence for the Riccati, Pantograph, and Beam Deformation Equations

In this section, three types of nonlinear equations, namely the Riccati equation, pantograph equation, and beam deformation equation, will be solved by the TAM and the convergence will be proved.

4.1 Riccati differential equation

Consider the following nonlinear Riccati differential equation [21].

$$u'(x) = A(x) + B(x)u(x) + C(x)u^2(x), \quad u(x_0) = \alpha, \quad x_0 \leq x \leq X_f \qquad (5)$$

where $A(x)$, $B(x)$ and $C(x)$ are continuous functions, x_0, X_f and α are arbitrary constants, and $u(x)$ is unknown function.

The Riccati differential equation is named after the Italian noble man Count Jacopo Francesco Riccati (1676-1754) [7]. This equation has many applications such as stochastic realization theory, robust stabilization, and network synthesis. Several applications are available in literature such as financial mathematics [8], control the boundary arising in fluid structure interaction falls in the class of SECS [26].

A substantial amount of research work has been done to develop the solution of Riccati differential equation. The most used methods are ADM [1] HAM [15, 34] Taylor matrix method [23] and Haar wavelet method [27], HPM [32] combination of Laplace, Adomian decomposition and Pade approximation methods [25], and many other methods available in literature. Two examples will be solved by TAM.

4.1. Example 1

Let us consider the following Riccati differential equation [37],

$$u' = u^2 - 2xu + x^2 + 1, u(0) = \frac{1}{2}. \qquad (6)$$

We apply the TAM by first distributing the equation as,

$$L(u) = u', \qquad N(u) = -u^2 + 2xu \quad \text{and } g(x) = -x^2 - 1. \qquad (7)$$

Thus, the initial problem which needs to be solved is

$$L(u_0(x)) = x^2 + 1, \qquad u_0(0) = \frac{1}{2}. \qquad (8)$$

By using simple manipulation, one can solve Eq. (8) as follows:

$$\int_0^x u_0'(x)\, dx = \int_0^x (x^2 + 1)\, dx, \ \ u_0(0) = \frac{1}{2}. \qquad (9)$$

Then, we get

$$u_0(x) = \frac{1}{2} + x + \frac{x^3}{3}. \qquad (10)$$

The second iteration can be carried through

$$L(u_1(x)) + N(u_0(x)) + g(x) = 0, \qquad u_1(0) = \frac{1}{2}. \qquad (11)$$

By integrating both sides of equation (11) from 0 to x, we get

$$\int_0^x u_1'(x)\, dx = \int_0^x (u_0^2(x) - 2xu_0(x) + x^2 + 1)\, dx, \quad u_1(0) = \frac{1}{2}. \qquad (12)$$

Thus,

$$u_1(x) = \frac{1}{2} + \frac{5x}{4} + \frac{x^4}{12} + \frac{x^7}{63}. \qquad (13)$$

The next iteration is

$$L(u_2(x)) + N(u_1(x)) + g(x) = 0, \qquad u_2(0) = \frac{1}{2}. \qquad (14)$$

By solving Eq. (14), we have

$$u_2(x) = \frac{1}{2} + \frac{5x}{4} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^5}{60} + \frac{x^6}{144} + \frac{x^8}{504} + \frac{5x^9}{3024} + \frac{x^{12}}{4536} + \frac{x^{15}}{59535}. \quad (15)$$

By continuing in this manner, we will get a series of the form:

$$u(x) = \lim_{n\to\infty} u_n(x) = \frac{1}{2} + \frac{5x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \frac{x^4}{32} + \frac{x^5}{64} + \frac{x^6}{128} + \frac{x^7}{256} + \frac{x^8}{512} + \cdots, \quad (16)$$

This series converges to the exact solution given in [37],

$$u(x) = x + \frac{1}{2-x}, \qquad |x| < 2 \quad (17)$$

$$= \frac{1}{2} + \frac{5x}{4} + \frac{x^2}{8} + \frac{x^3}{16} + \frac{x^4}{32} + \frac{x^5}{64} + \frac{x^6}{128} + \frac{x^7}{256} + \frac{x^8}{512} + \frac{x^9}{1024} + \frac{x^{10}}{2048} + \cdots.$$

Suppose that $N: [0,1] \to R$, then $u_n = N(u_{n-1})$ and $0 \le x \le 1$.

According to the Theorem 3.3 for nonlinear mapping N, a sufficient condition for convergence of the TAM is strictly contraction N, the exact solution is $u = u(x) = x + \frac{1}{2-x}$, $|x| < 2$, therefore, we have

$$\|u_0 - u\| = \left\|\frac{1}{2} + x + \frac{x^3}{3} - \left(x + \frac{1}{2-x}\right)\right\|,$$

$$\|u_1 - u\| = \left\|\frac{1}{2} + \frac{5x}{4} + \frac{x^4}{12} + \frac{x^7}{63} - \left(x + \frac{1}{2-x}\right)\right\|$$

$$\le \left\|\frac{1}{2} + x + \frac{x^3}{3} - \left(x + \frac{1}{2-x}\right)\right\| \left\|\left(\frac{\frac{1}{2} + \frac{5x}{4} + \frac{x^4}{12} + \frac{x^7}{63} - \left(x + \frac{1}{2-x}\right)}{\frac{1}{2} + x + \frac{x^3}{3} - \left(x + \frac{1}{2-x}\right)}\right)\right\|,$$

But, $\forall\, x \in [0,1]$, $0 < k < 1$, When $x = \frac{1}{2}$,

$$\left\|\left(\frac{\frac{1}{2} + \frac{5x}{4} + \frac{x^4}{12} + \frac{x^7}{63} - \left(x + \frac{1}{2-x}\right)}{\frac{1}{2} + x + \frac{x^3}{3} - \left(x + \frac{1}{2-x}\right)}\right)\right\| \le k = 0.290675 < 1, \text{ thus,}$$

$$\|u_1 - u\| \le k \left\|\frac{1}{2} + x + \frac{x^3}{3} - \left(x + \frac{1}{2-x}\right)\right\| = k\,\|u_0 - u\|,$$

Then, we get

$$\|u_1 - u\| \le k\,\|u_0 - u\|,$$

$\|u_2 - u\| = \Big\|\frac{1}{2} + \frac{5x}{4} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^5}{60} + \frac{x^6}{144} + \frac{x^8}{504} + \frac{5x^9}{3024} +$
x124536+x1559535−(x+12−x)≤(12+5x4+x412+x763−x+12−x)(12+5x4+x28+x348+x560+x6144+x8504+5x93024+x124536+x1559535−x+12−x12+5x4+x412+x763−x+12−x)Since, ∀ x ϵ 0,1, 0<k<1,When x= 12,

$$\left\|\left(\frac{\frac{1}{2} + \frac{5x}{4} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^5}{60} + \frac{x^6}{144} + \frac{x^8}{504} + \frac{5x^9}{3024} + \frac{x^{12}}{4536} + \frac{x^{15}}{59535} - \left(x + \frac{1}{2-x}\right)}{\frac{1}{2} + \frac{5x}{4} + \frac{x^4}{12} + \frac{x^7}{63} - \left(x + \frac{1}{2-x}\right)}\right)\right\| \le k = 0.197393 < 1,$$

$$\|u_2 - u\| \le k \left\|\frac{1}{2} + \frac{5x}{4} + \frac{x^4}{12} + \frac{x^7}{63} - \left(x + \frac{1}{2-x}\right)\right\| = k\|u_1 - u\| \le kk\|u_0 - u\|,$$

Thus,

$$\|u_2 - u\| \le k^2\|u_0 - u\|,$$

Similarly, we have

$$\|u_3 - u\| \le k^3\|u_0 - u\|,$$

By continuing in this way we get:

$$\|u_n - u\| \le k^n\|u_0 - u\|,$$

Therefore,

$$\lim_{n\to\infty}\|u_n - u\| \le \lim_{n\to\infty} k^n\,\|u_0 - u\| = 0,$$

$\lim_{n\to\infty} k^n = 0$, then

$\lim_{n\to\infty}\|u_n - u\| = 0 \to \lim_{n\to\infty} u_n = u$, that is $u(x) = \lim_{n\to\infty} u_n(x) = x + \frac{1}{2-x}$, which is an exact solution.

4.1. Example 2

Let us consider the following Riccati differential equation [22]

$$u' = e^x - e^{3x} + 2e^{2x}u - e^x u^2, \qquad u(0) = 1. \quad (18)$$

Applying TAM as earlier by first distributing the equation as,

$$L(u) = u', \qquad N(u) = -2e^{2x}u + e^x u^2 \quad \text{and} \quad g(x) = -e^x + e^{3x}. \quad (19)$$

Thus, the initial problem will be

$$L(u_0(x)) = e^x - e^{3x}, \qquad u_0(0) = 1. \quad (20)$$

By integrating both sides of equation (20) from 0 to x, we obtain

$$\int_0^x u_0'(x)\,dx = \int_0^x (e^x - e^{3x})\,dx, \quad u_0(0) = 1 \quad (21)$$

Therefore, we have

$$u_0(x) = \frac{1}{3} + e^x - \frac{e^{3x}}{3}. \quad (22)$$

The second iteration can be given as

$$L(u_1(x)) + N(u_0(x)) + g(x) = 0, \qquad u_1(0) = 1. \quad (23)$$

By integrating both sides of equation (23) from 0 to x, we get

$$\int_0^x u_1'(x)\,dx = \int_0^x \left(e^x - e^{3x} + 2e^{2x}u_0(x) - e^x u_0^2(x)\right) dx, \quad u_1(0) = 1. \quad (24)$$

Then, we obtain

$$u_1(x) = \frac{1}{14} + \frac{8e^x}{9} + \frac{e^{4x}}{18} - \frac{e^{7x}}{63}. \quad (25)$$

We turn the function $u_1(x)$ by using Taylor series expansion to exponential function, we get

$$u_1(x) = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{23x^4}{24} - \frac{209x^5}{120}. \quad (26)$$

Applying the same process, we get the second iteration $u_2(x)$,

$$L(u_2(x)) + N(u_1(x)) + g(x) = 0, \qquad u_2(0) = 1. \quad (27)$$

Then, we have

$$u_2(x) = \frac{11}{9720} + \frac{195e^x}{196} + \frac{e^{2x}}{126} - \frac{e^{3x}}{243} - \frac{e^{5x}}{630} + \frac{e^{6x}}{486} + \frac{e^{8x}}{3528} - \frac{5e^{9x}}{6804} + \frac{e^{12x}}{6804} - \frac{e^{15x}}{59535}. \quad (28)$$

We turn the function $u_2(x)$ to series as following

$$u_2(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} - \frac{79x^7}{5040} - \frac{3919x^8}{40320} - \frac{116479x^9}{362880}. \quad (29)$$

By continuing in this way we will get a series of the form:

$$u(x) = \lim_{n\to\infty} u_n(x)$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \cdots. \quad (30)$$

This series converges to the following exact solution [22]

$$u(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \cdots. \quad (31)$$

Suppose that $N:[0,1] \to R$, then $u_n = N(u_{n-1})$ and $0 \le x \le 1$.

Similar steps as for example 1 can be followed, since the exact solution is $u = u(x) = e^x$, therefore, we have

$$\|u_0 - u\| = \left\|\frac{1}{3} + e^x - \frac{e^{3x}}{3} - e^x\right\|,$$

$$\|u_1 - u\| = \left\|1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{23x^4}{24} - \frac{209x^5}{120} - e^x\right\|$$

$$\le \left\|(\frac{1}{3} + e^x - \frac{e^{3x}}{3} - e^x)\right\| \left\|(\frac{1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{23x^4}{24} - \frac{209x^5}{120} - e^x}{\frac{1}{3} + e^x - \frac{e^{3x}}{3} - e^x})\right\|,$$

But, $\forall\, x \in [0,1]$, $0 < k < 1$, When $x = \frac{1}{3}$,

$$\left\|(\frac{1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{23x^4}{24} - \frac{209x^5}{120} - e^x}{\frac{1}{3} + e^x - \frac{e^{3x}}{3} - e^x})\right\| \le k = 0.0556864 < 1, \text{ thus,}$$

$$\|u_1 - u\| \le k \left\|\frac{1}{3} + e^x - \frac{e^{3x}}{3} - e^x\right\| = k\,\|u_0 - u\|,$$

Then, we have

$$\|u_1 - u\| \le k\,\|u_0 - u\|,$$

$$\|u_2 - u\| = \left\|\frac{11}{9720} + \frac{195e^x}{196} + \frac{e^{2x}}{126} - \frac{e^{3x}}{243} - \frac{e^{5x}}{630} + \frac{e^{6x}}{486} + \frac{e^{8x}}{3528} -\right.$$
5ℯ9x6804+ℯ12x6804−ℯ15x59535−ℯx≤(1+x+x22−x36−23x424−209x5120−ℯx)(119720+195ℯx196+ℯ2x126−ℯ3x243−ℯ5x630+ℯ6x486+ℯ8x3528−5ℯ9x6804+ℯ12x6804−ℯ15x59535−ℯx1+x+x22−x36−23x424−209x5120−ℯx)Since, ∀ x ϵ 0,1, 0<k<1,When x= 13,

$$\left\|(\frac{\frac{11}{9720} + \frac{195e^x}{196} + \frac{e^{2x}}{126} - \frac{e^{3x}}{243} - \frac{e^{5x}}{630} + \frac{e^{6x}}{486} + \frac{e^{8x}}{3528} - \frac{5e^{9x}}{6804} + \frac{e^{12x}}{6804} - \frac{e^{15x}}{59535} - e^x}{1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{23x^4}{24} - \frac{209x^5}{120} - e^x})\right\| \le k = 0.00207751 < 1,$$

$$\|u_2 - u\| \le k\left\|1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{23x^4}{24} - \frac{209x^5}{120} - e^x\right\| = k\|u_1 - u\| \le kk\|u_0 - u\|,$$

Thus, we get

$$\|u_2 - u\| \le k^2\|u_0 - u\|,$$

Similarly, we have

$$\|u_3 - u\| \le k^3\|u_0 - u\|,$$

By continuing in this way we get:

$$\|u_n - u\| \le k^n\|u_0 - u\|,$$

Therefore,

$\lim_{n\to\infty}\|u_n - u\| \le \lim_{n\to\infty} k^n\,\|u_0 - u\| = 0$, $\lim_{n\to\infty} k^n = 0$, then

$\lim_{n\to\infty}\|u_n - u\| = 0 \to \lim_{n\to\infty} u_n = u$, that is $u(x) = \lim_{n\to\infty} u_n(x) = e^x$,

which is an exact solution.

4.2 Pantograph differential equation

Pantograph equation used in many applications, such as industrial applications [20], studies based on biology, economy, control and electrodynamics [14].

Pantograph equation was solved by many authors either analytically or numerically. For instance Yang and Huang presented a spectral-collocation method for fractional pantograph delay-integro differential equations [39], [40] proposed an efficient algorithm for solving generalized pantograph equations with linear functional argument, the authors of investigated an exponential approximation to obtain an approximate solution of generalized pantograph-delay differential equations [41]. In [36] the authors proposed a new collocation scheme based on Bernoulli operational matrix for numerical solution of generalized pantograph equation. Recently, Doha et al. proposed and developed a new Jacobi rational-Gauss collocation method for solving the generalized pantograph equations on a semi-infinite domain [18]. Two examples of pantograph equations will be solved by TAM.

4.2. Example 3:

Consider the following pantograph differential equation [33],

$$u' = \frac{1}{2}u + \frac{1}{2}e^{\frac{x}{2}}u\left(\frac{x}{2}\right), \quad u(0) = 1. \quad (32)$$

By implementing the TAM:

$$L(u) = u', \quad N(u) = -\frac{1}{2}u - \frac{1}{2}e^{\frac{x}{2}}u\left(\frac{x}{2}\right) \quad \text{and} \quad g(x) = 0. \quad (33)$$

By first distributing equation as,

$$L\big(u_0(x)\big) = 0, \quad u_0(0) = 1. \quad (34)$$

By solving Eq.(34), we have

$$\int_0^x u_0'(x)\,dx = 0, \quad u_0(0) = 1, \quad (35)$$

Then, we get

$$u_0(x) = 1, \quad (36)$$

The second iteration will be

$$L\big(u_1(x)\big) + N\big(u_0(x)\big) + g(x) = 0, \quad u_1(0) = 1. \quad (37)$$

By integrating both side of problem (37),

$$\int_0^x u_1'(x)\,dx = \int_0^x \left(\frac{1}{2}u_0(x) + \frac{1}{2}e^{\frac{x}{2}}u_0(x)\right)dx,$$
$$u_1(0) = 1. \quad (38)$$

Then, we obtain

$$u_1(x) = e^{\frac{x}{2}} + \frac{x}{2}. \quad (39)$$

The next iteration is

$$L\big(u_2(x)\big) + N\big(u_1(x)\big) + g(x) = 0, \quad u_2(0) = 1. \quad (40)$$

Therefore, we have

$$u_2(x) = -\frac{1}{6} + \frac{e^{\frac{x}{2}}}{2} + \frac{2}{3}e^{\frac{3x}{4}} + \frac{1}{4}x\,e^{\frac{x}{2}} + \frac{x^2}{8}. \quad (41)$$

We turn the function $u_2(x)$ by using Taylor series expansion to exponential function,

$$u_2(x) = 1 + x + \frac{x^2}{2} + \frac{17x^3}{192} + \frac{47x^4}{3072} + \frac{43x^5}{20480} + \frac{71x^6}{294912} + \frac{197x^7}{8257536}. \quad (42)$$

The next iteration is

$$L\big(u_3(x)\big) + N\big(u_2(x)\big) + g(x) = 0, \quad u_3(0) = 1. \quad (43)$$

In similar manner, we get

$$u_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{377x^4}{12288} + \frac{2479x^5}{491520} + \frac{16109x^6}{23592960} + \frac{104267x^7}{1321205760}. \quad (44)$$

By continuing in this way we will get the following series:

$$u(x) = \lim_{n\to\infty} u_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \cdots. \quad (45)$$

This series converges to the exact solution [33],

$$u(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \cdots. \quad (46)$$

Suppose that $N: [0,1] \to R$, then $u_n = N(u_{n-1})$ and $0 \le x \le 1$.

Similar procedure can be followed; the exact solution is $u = u(x) = e^x$, therefore we have

$$\|u_0 - u\| = \|1 - e^x\|,$$

$$\|u_1 - u\| = \left\|e^{\frac{x}{2}} + \frac{x}{2} - e^x\right\| \le \|(1 - e^x)\| \left\|\left(\frac{e^{\frac{x}{2}} + \frac{x}{2} - e^x}{1 - e^x}\right)\right\|,$$

But, $\forall\, x \in [0,1]$, $0 < k < 1$, When $x = \frac{1}{3}$, $\left\|\left(\frac{e^{\frac{x}{2}} + \frac{x}{2} - e^x}{1 - e^x}\right)\right\| \le$ $k = 0.120283 < 1$, thus,

$$\|u_1 - u\| \le k\|1 - e^x\| = k\,\|u_0 - u\|,$$

Then, we have

$$\|u_1 - u\| \le k\,\|u_0 - u\|,$$

$$\|u_2 - u\| = \left\|-\frac{1}{6} + \frac{e^{\frac{x}{2}}}{2} + \frac{2}{3}e^{\frac{3x}{4}} + \frac{1}{4}x\, e^{\frac{x}{2}} + \frac{x^2}{8} - e^x\right\| \le \left\|\left(e^{\frac{x}{2}} + \frac{x}{2} - e^x\right)\right\| \left\|\left(\frac{-\frac{1}{6} + \frac{e^{\frac{x}{2}}}{2} + \frac{2}{3}e^{\frac{3x}{4}} + \frac{1}{4}x\, e^{\frac{x}{2}} + \frac{x^2}{8} - e^x}{e^{\frac{x}{2}} + \frac{x}{2} - e^x}\right)\right\|,$$

Since, $\forall\, x \in [0,1]$, $0 < k < 1$, When $x = \frac{1}{3}$,

$$\left\|\left(\frac{-\frac{1}{6} + \frac{e^{\frac{x}{2}}}{2} + \frac{2}{3}e^{\frac{3x}{4}} + \frac{1}{4}x\, e^{\frac{x}{2}} + \frac{x^2}{8} - e^x}{e^{\frac{x}{2}} + \frac{x}{2} - e^x}\right)\right\| \le k = 0.0682217 < 1,$$

$$\|u_2 - u\| \le k\left\|e^{\frac{x}{2}} + \frac{x}{2} - e^x\right\| = k\|u_1 - u\| \le kk\|u_0 - u\|,$$

Thus, we get

$$\|u_2 - u\| \le k^2\|u_0 - u\|,$$

Similarly, we have

$$\|u_3 - u\| \le k^3\|u_0 - u\|,$$

By continuing in this way we get:

$$\|u_n - u\| \le k^n\|u_0 - u\|,$$

Therefore,

$$\lim_{n\to\infty}\|u_n - u\| \le \lim_{n\to\infty}k^n\,\|u_0 - u\| = 0,$$

$\lim_{n\to\infty}k^n = 0$, then

$\lim_{n\to\infty}\|u_n - u\| = 0 \to \lim_{n\to\infty} u_n = u$, that is $u(x) = \lim_{n\to\infty} u_n(x) = e^x$, which is an exact solution.

4.2. Example 4

Let us deal with the following pantograph differential equation [33],

$$u'' = \frac{3}{4}u + u\left(\frac{x}{2}\right) - x^2 + 2,$$
$$u(0) = 0\,, u'(0) = 0. \quad (47)$$

We apply the TAM by first distributing the equation as,

$$L(u) = u'', \quad N(u) = -\frac{3}{4}u - u\left(\frac{x}{2}\right)$$
$$\text{and} \quad g(x) = x^2 - 2. \quad (48)$$

Thus, the initial problem which needs to be solved is

$$L(u_0(x)) = -x^2 + 2,$$
$$u_0(0) = 0 \text{ and } u_0'(0) = 0. \quad (49)$$

By integrating both sides of problem (49), will be achieved

$$\int_0^x u_0''(x)\,dx = \int_0^x (-x^2 + 2)dx\,,$$
$$u_0(0) = 0 \text{ and } u_0'(0) = 0. \quad (50)$$

Then, we have

$$u_0'(x) = -\frac{x^3}{3} + 2x, \qquad u_0(0) = 0. \quad (51)$$

Once again, by taking the integration to both sides of problem (51), we have

$$\int_0^x u_0'(x)\,dx = \int_0^x \left(-\frac{x^3}{3} + 2x\right)dx,$$
$$u_0(0) = 0. \quad (52)$$

Then, we obtain

$$u_0(x) = -\frac{x^4}{12} + x^2. \quad (53)$$

The second iteration can be carried as

$$L(u_1(x)) + N(u_0(x)) + g(x) = 0,$$
$$u_1(0) = 0 \text{ and } u_1'(0) = 0. \quad (54)$$

By integrating both sides of equation (54) from 0 to x, we obtain

$$\int_0^x\int_0^x u_1''(x)\,dxdx = \int_0^x\int_0^x \left(-x^2 + 2 + \frac{3}{4}u_0(x) + u_0\left(\frac{x}{2}\right)\right)dxdx,$$
$$u_1(0) = 0, \qquad u_1'(0) = 0 \quad (55)$$

Thus,

$$u_1(x) = x^2 - \frac{13x^6}{5760}. \quad (56)$$

The next iteration is

$$L(u_2(x)) + N(u_1(x)) + g(x) = 0\,,$$
$$u_2(0) = 0 \text{ and } u_2'(0) = 0. \quad (57)$$

Thus, we have

$$u_2(x) = x^2 - \frac{91x^8}{2949120}. \quad (58)$$

The next iteration is

$$L(u_3(x)) + N(u_2(x)) + g(x) = 0\,,$$
$$u_3(0) = 0 \text{ and } u_3'(0) = 0. \quad (59)$$

Then, we get

$$u_3(x) = x^2 - \frac{17563x^{10}}{67947724800}. \quad (60)$$

By continuing in this way we will get:

$$u_n(x) = x^2 - \text{small term}. \quad (61)$$

This series converges to the exact solution [33],

$$u(x) = \lim_{n\to\infty} u_n(x) = x^2. \quad (62)$$

Suppose that $N: [0,1] \to R$, then $u_n = N(u_{n-1})$ and $0 \le x \le 1$.

Following the same procedure, the exact solution is $u = u(x) = x^2$, therefore we have

$$\|u_0 - u\| = \left\|-\frac{x^4}{12} + x^2 - (x^2)\right\|,$$

$$\|u_1 - u\| = \left\| x^2 - \frac{13x^6}{5760} - (x^2) \right\|$$
$$\leq \left\| (-\frac{x^4}{12} + x^2 - (x^2)) \right\| \left\| (\frac{x^2 - \frac{13x^6}{5760} - (x^2)}{-\frac{x^4}{12} + x^2 - (x^2)}) \right\|,$$

But, $\forall\, x \in [0,1]$, $0 < k < 1$, When $x = \frac{1}{4}$,

$\left\| (\frac{x^2 - \frac{13x^6}{5760} - (x^2)}{-\frac{x^4}{12} + x^2 - (x^2)}) \right\| \leq k = 0.00169271 < 1$, thus,

$\|u_1 - u\| \leq k \left\| -\frac{x^4}{12} + x^2 - (x^2) \right\| = k\,\|u_0 - u\|$,

Then, we have

$\|u_1 - u\| \leq k\,\|u_0 - u\|$,

$\|u_2 - u\| = \left\| x^2 - \frac{91x^8}{2949120} - (x^2) \right\| \leq \left\| (x^2 - \frac{13x^6}{5760} -$
(x2))(x2−91x82949120−(x2)x2−13x65760−(x2)),

Since, $\forall\, x \in [0,1]$, $0 < k < 1$, When $x = \frac{1}{4}$,

$\left\| (\frac{x^2 - \frac{91x^8}{2949120} - (x^2)}{x^2 - \frac{13x^6}{5760} - (x^2)}) \right\| \leq k = 0.000854492 < 1$,

$\|u_2 - u\| \leq k \left\| x^2 - \frac{13x^6}{5760} - (x^2) \right\| = k\|u_1 - u\| \leq kk\|u_0 - u\|$,

Thus, we get

$\|u_2 - u\| \leq k^2\|u_0 - u\|$,

Similarly, we have

$\|u_3 - u\| \leq k^3\|u_0 - u\|$,

By continuing in this way we get:

$\|u_n - u\| \leq k^n\|u_0 - u\|$,

Therefore,

$\lim_{n\to\infty}\|u_n - u\| \leq \lim_{n\to\infty} k^n\,\|u_0 - u\| = 0$,

$\lim_{n\to\infty} k^n = 0$, then

$\lim_{n\to\infty}\|u_n - u\| = 0 \rightarrow \lim_{n\to\infty} u_n = u$, that is $u(x) = \lim_{n\to\infty} u_n(x) = x^2$, which is an exact solution.

4.3 Beam differential equation

According to the classical beam theory, the function $u = u(x)$ represents the configuration of the deformed beam. The length of the elastic beam is L =1, where x =0 at the left side and x =1 at the right side and f =f(x) is a load which causes the deformation [10], as it is shown in Figures (1) and (2).

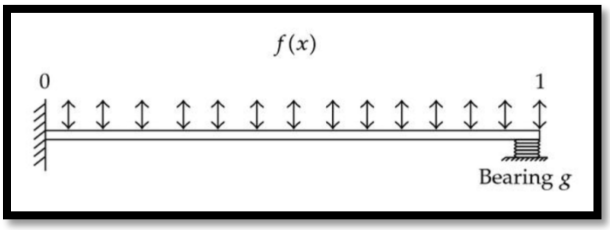

Figure1: Beam on elastic bearing.

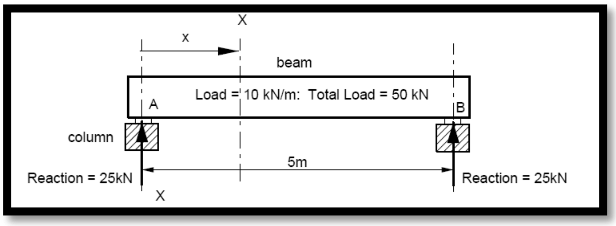

Figure 2: Loads and Reactions on a simply supported beam

Let us consider the nonlinear beam deformation problem as a general fourth order boundary value problem of the form [10],

$$u''''(x) = f(x, u, u', u'', u''') \qquad (63)$$

with the boundary conditions:

$$u(a) = \alpha_1, \qquad u'(a) = \alpha_2,$$
$$u(b) = \beta_1, \qquad u'(b) = \beta_2,$$

where f is a continuous function on [a, b] and the parameters α_1, α_2, β_1 and β_2 are finite real arbitrary constants. Eq. (63) used as mathematical models in viscoelastic and in elastic flows [30], deformation of beams [29] and plate deflection theory [16].

Recently, many analytical methods are used to solve nonlinear elastic beam deformation problems, such as HPM [3], the VIM [10], and the ADM [2]. The following two problems of the beam deformation will be solved by TAM.

4.3. Example 5

Consider the following form of the beam deformation equation [10]

$u'''' = u^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48,$

$$u(0) = 0, u'(0) = 0, u(1) = 1, u'(1) = 1. \qquad (64)$$

Applying TAM as earlier by first distributing the equation,

$$L(u) = u'''', N(u) = -u^2,$$
$$g(x) = x^{10} - 4x^9 + 4x^8 + 4x^7 - 8x^6 + 4x^4 - 120x + 48. \qquad (65)$$

Thus, the initial problem which needs to be solved is

$$L(u_0(x)) + g(x) = 0\ ,$$
$$u_0(0) = 0, u_0{}'(0) = 0, u_0(1) = 1, u_0{}'(1) = 1 \qquad (66)$$

By integrating both sides of (66) from 0 to x four times, one can obtain

$$\int_0^x\int_0^x\int_0^x\int_0^x u_0{}''''(x)\,dx\,dx\,dx\,dx = \int_0^x\int_0^x\int_0^x\int_0^x -g(x)\,dx\,dx\,dx\,dx\,,$$
$$u_0(0) = 0, u_0{}'(0) = 0, u_0(1) = 1, u_0{}'(1) = 1. \qquad (67)$$

Then, we have

$$u_0(x) = \frac{718561x^2}{360360} + \frac{4019x^3}{540540} - 2x^4 + x^5 - \frac{x^8}{420} + \frac{x^{10}}{630} - \frac{x^{11}}{1980} - \frac{x^{12}}{2970} + \frac{x^{13}}{4290} - \frac{x^{14}}{24024}. \qquad (68)$$

The second iteration can be carried through and we have

$$L(u_1(x)) + N(u_0(x)) + g(x) = 0,$$
$$u_1(0) = 0, u_1{}'(0) = 0, u_1(1) = 1, u_1{}'(1) = 1. \qquad (69)$$

Once again, by taking the integration to both sides of problem (69) four times from 0 to x, one can have:

$$\int_0^x\int_0^x\int_0^x\int_0^x u_1''''(x)\,dx\,dx\,dx\,dx = \int_0^x\int_0^x\int_0^x\int_0^x ((u_0(x))^2 - g(x))\,dx\,dx\,dx\,dx,$$

$$u_1(0)=0, u_1'(0)=0, u_1(1)=1, u_1'(1)=1. \quad (70)$$

Thus, we get,

$$u_1(x) = \frac{472079268428230850536735 9x^2}{236041030958866189056000 0} + \frac{1855324832786792 6839x^3}{118020515479433094528000 0} - 2x^4 + x^5 - \frac{3107407679x^8}{218163673728000} + \frac{2887896659x^9}{294520959532800} + \frac{7018307521x^{10}}{1472604797664000} - \frac{22553x^{11}}{4281076800} + \frac{4019x^{12}}{3210807600} - \frac{718561x^{14}}{1818030614400} - \frac{4019x^{15}}{3718698984000} + \frac{1799641x^{16}}{4958265312000} - \frac{10117717x^{17}}{85587287385600} - \frac{68671x^{18}}{655004750400} + \frac{127713533x^{19}}{1941434080185600} + \frac{11676571x^{20}}{7550021422944000} - \frac{2086087x^{21}}{186529941037440} + \frac{11827x^{22}}{3321402084000} - \frac{19x^{23}}{49945896000} + \frac{x^{24}}{61856071200} - \frac{x^{25}}{111891780000} - \frac{1153x^{26}}{672022030680000} + \frac{289x^{27}}{112699346760000} - \frac{2857x^{28}}{5522267991240000} - \frac{41x^{29}}{203359710153600} + \frac{191x^{30}}{1525197826152000} - \frac{x^{31}}{38914512436800} + \frac{x^{32}}{498105759191040}. \quad (71)$$

The next iteration is

$$L(u_2(x)) + N(u_1(x)) + g(x) = 0,$$

$$u_2(0)=0, u_2'(0)=0, u_2(1)=1, u_2'(1)=1 \quad (72)$$

By solving Eq. (72), we get

$$u_2(x) = \frac{2202693265997560554567213964153742646845680934915346589114693562189521\ x^2}{110134664427266869792315114409057766130380888118760359015219200000000} + \frac{85563266301725606658409388569749904382924510090028256764730 1\ x^3}{311701880454528876770703153987899338104851570147434978344960000000000} - 2x^4 + x^5 - \frac{26375047640697571401725850482950314285736 5119\ x^8}{93601818737489036878394910509624913407508480000000000} + \frac{87586038975871882144948292891844699430648201\ x^9}{421208184318700665952777097293312 1\ 10333788160000000 0} + \frac{659381584143573226578609977219716768254919 21\ x^{10}}{70201364053116777658796182882218685055631360000000000} + O[x]^{11}. \quad (73)$$

In order to check the accuracy of the approximate solution, we calculate the absolute error, $|r_n| = |u(x) - u_n(x)|$ where $u(x)=x^5-2x^4+2x^2$ is the exact solution and $u_n(x)$ is the approximate solution. In Table (1) we compare the absolute error of TAM with n=2 together with the HPM, and the VIM.

Table 1: Comparison of the absolute errors for TAM, VIM and HPM

x	$\lvert r_2\rvert$ for TAM	$\lvert r_2\rvert$ for VIM	$\lvert r_2\rvert$ for HPM
0	0	0	0
0.1	2.0×10^{-10}	6.8×10^{-9}	1.896×10^{-7}
0.2	6.0×10^{-10}	2.39×10^{-8}	6.4171×10^{-7}
0.3	1.1×10^{-9}	4.64×10^{-8}	1.18180×10^{-6}
0.4	1.5×10^{-9}	6.92×10^{-8}	1.6405×10^{-6}
0.5	1.8×10^{-9}	8.74×10^{-8}	1.8703×10^{-6}
0.6	1.7×10^{-9}	9.61×10^{-8}	1.7815×10^{-6}
0.7	1.3×10^{-9}	9.06×10^{-8}	1.3816×10^{-6}
0.8	8.0×10^{-10}	6.7×10^{-8}	7.958×10^{-7}
0.9	2.0×10^{-10}	2.92×10^{-8}	2.437×10^{-7}
1.0	0	1.2×10^{-9}	6.0×10^{-10}

It can be observed clearly from Table (1), the absolute error for TAM is lower than VIM and HPM.

Suppose that $N: [0,1] \rightarrow R$, then $u_n = N(u_{n-1})$ and $0 \le x \le 1$.

The convergence issue can be done as follows:

Since, the exact solution is $u = u(x) = x^5 - 2x^4 + 2x^2$, therefore, we have

$\|u_0 - u\| = \|u_0 - (x^5 - 2x^4 + 2x^2)\|,$

$\|u_1 - u\| = \|u_1 - (x^5 - 2x^4 + 2x^2)\| \le \|(u_0 - (x^5 - 2x^4 + 2x^2))\| \left\|\left(\frac{u_1 - (x^5 - 2x^4 + 2x^2)}{u_0 - (x^5 - 2x^4 + 2x^2)}\right)\right\|,$

But, $\forall\, x \in [0,1]$, $0 < k < 1$, When $x = \frac{1}{2}$,

$\left\|\left(\frac{u_1-(x^5-2x^4+2x^2)}{u_0-(x^5-2x^4+2x^2)}\right)\right\| \le k = 0.00178287 < 1$, thus,

$\|u_1 - u\| \le k\|u_0 - (x^5 - 2x^4 + 2x^2)\| = k\,\|u_0 - u\|,$

Then, we have

$\|u_1 - u\| \le k\,\|u_0 - u\|,$

$\|u_2 - u\| = \|u_2 - (x^5 - 2x^4 + 2x^2)\| \le \|(u_1 - (x^5 - 2x4+2x2))(u2-(x5-2x4+2x2)u1-(x5-2x4+2x2))$Since, $\forall\, x \in [0,1]$, $0 < k < 1$, When $x = \frac{1}{2}$,

$\left\|\left(\frac{u_2-(x^5-2x^4+2x^2)}{u_1-(x^5-2x^4+2x^2)}\right)\right\| \le k = 0.00170467 < 1,$

$\|u_2 - u\| \le k\|u_1 - (x^5 - 2x^4 + 2x^2)\| = k\|u_1 - u\| \le kk\|u_0 - u\|,$

Thus, we get

$\|u_2 - u\| \le k^2\|u_0 - u\|,$

Similarly, we have

$\|u_3 - u\| \le k^3\|u_0 - u\|,$

By continuing in this way we get:

$\|u_n - u\| \le k^n\|u_0 - u\|,$

Therefore,

$\lim_{n\to\infty}\|u_n - u\| \le \lim_{n\to\infty} k^n\,\|u_0 - u\| = 0,$

$\lim_{n\to\infty} k^n = 0$, then

$\lim_{n\to\infty}\|u_n - u\| = 0 \rightarrow \lim_{n\to\infty} u_n = u$, that is $u(x) = \lim_{n\to\infty} u_n(x) = x^5 - 2x^4 + 2x^2$, which is an exact solution.

4.3. Example 6

Consider the beam deformation equation [11],

$$u'''' = u + u'' + e^x(x-3),$$

$$u(0)=1, u'(0)=0, u(1)=0, u'(1)=-e. \quad (74)$$

In the following, the TAM will be used,

$$L(u) = u'''', \quad N(u) = -u - u'',$$

$$g(x) = -e^x(x-3). \quad (75)$$

Applying TAM as earlier by first distributing the equation like

$$L\big(u_0(x)\big) + g(x) = 0\,,$$
$$u_0(0) = 1, u_0'(0) = 0, u_0(1) = 0, u_0'(1) = -e. \quad (76)$$

By integrating both sides of equation (76) from 0 to x, we will have

$$\int_0^x\int_0^x\int_0^x\int_0^x (u_0''''(x))\,dx\,dx\,dx\,dx = \int_0^x\int_0^x\int_0^x\int_0^x e^x(x-3)\,dx\,dx\,dx\,dx\,,$$
$$u_0(0) = 1, u_0'(0) = 0, u_0(1) = 0, u_0'(1) = -e. \quad (77)$$

Then, we get

$$u_0(x) = 8 - 7e^x + 6x + e^x x - 36x^2 + 14ex^2 + 22x^3 - 8ex^3. \quad (78)$$

The second iteration can be carried through and is given as

$$L\big(u_1(x)\big) + N\big(u_0(x)\big) + g(x) = 0,$$
$$u_1(0) = 0, u_1'(0) = 0, u_1(1) = 2e, u_1'(1) = -e. \quad (79)$$

By integrating both sides of (79), it will be achieved

$$\int_0^x\int_0^x\int_0^x\int_0^x (u_1''''(x))\,dx\,dx\,dx\,dx = \int_0^x\int_0^x\int_0^x\int_0^x (e^x(x-3) + u_0(x) + u_0''(x))\,dx\,dx\,dx\,dx\,,$$
$$u_1(0) = 1, u_1'(0) = 0, u_1(1) = 0, u_1'(1) = -e. \quad (80)$$

Therefore, we have

$$u_1(x) = 28 - 27e^x + 24x + 3e^x x - \frac{13919x^2}{105} + \frac{22027ex^2}{420} + \frac{8626x^3}{105} - \frac{9211ex^3}{315} - \frac{8x^4}{3} + \frac{7ex^4}{6} + \frac{23x^5}{20} - \frac{2ex^5}{5} - \frac{x^6}{10} + \frac{7ex^6}{180} + \frac{11x^7}{420} - \frac{ex^7}{105}. \quad (81)$$

The next iteration is

$$L\big(u_2(x)\big) + N\big(u_1(x)\big) + g(x) = 0,$$
$$u_2(0) = 1, u_2'(0) = 0, u_2(1) = 0, u_2'(1) = -e. \quad (82)$$

Then, it can be obtained

$$u_2(x) = 80 - 79e^x + 72x + 7e^x x - \frac{12771847x^2}{33075} + \frac{10745568371ex^2}{69854400} + \frac{317593x^3}{1323} - \frac{1482272317ex^3}{17463600} - \frac{12449x^4}{1260} + \frac{22027ex^4}{5040} + \frac{4523x^5}{1050} - \frac{9211ex^5}{6300} - \frac{17279x^6}{37800} + \frac{27907ex^6}{151200} + \frac{11041x^7}{88200} - \frac{11731ex^7}{264600} - \frac{17x^8}{5040} + \frac{ex^8}{720} + \frac{x^9}{1344} - \frac{ex^9}{3780} - \frac{x^{10}}{50400} + \frac{ex^{10}}{129600} + \frac{x^{11}}{302400} - \frac{ex^{11}}{831600}. \quad (83)$$

The accuracy of the approximate solution has been proved by calculating the absolute error, where $u(x) = (1-x)e^x$ is the exact solution. In Table (2) we compare the absolute error of TAM for two iterations together with the VIM.

Table 2: Comparison of the absolute error for TAM and VIM

x	$\lvert r_2\rvert$ for TAM	$\lvert r_2\rvert$ for VIM
0	0	0
0.1	2.22945×10^{-7}	3.93180×10^{-6}
0.2	8.20857×10^{-7}	1.35716×10^{-5}
0.3	1.57987×10^{-6}	2.57244×10^{-5}
0.4	2.21818×10^{-6}	3.72912×10^{-5}
0.5	2.49333×10^{-6}	4.52445×10^{-5}
0.6	2.29399×10^{-6}	4.70100×10^{-5}
0.7	1.68701×10^{-6}	4.08379×10^{-5}
0.8	9.01995×10^{-7}	2.70944×10^{-5}
0.9	2.50721×10^{-7}	9.81890×10^{-6}
1.0	2.96245×10^{-14}	0

It can be seen clearly from Table (2), the absolute error for TAM is lower than for VIM.

Suppose that $N: [0,1] \to R$, then $u_n = N(u_{n-1})$ and $0 \le x \le 1$.

Finally, the convergence is given in the following steps, since the exact solution is $u = u(x) = (1-x)e^x$, therefore, we have

$\|u_0 - u\| = \|u_0 - (1-x)e^x\|$,

$\|u_1 - u\| = \|u_1 - (1-x)e^x\| \le \|(u_0 - (1-x)e^x)\| \left\|\left(\frac{u_1 - (1-x)e^x}{u_0 - (1-x)e^x}\right)\right\|$,

But, $\forall x \in [0,1]$, $0 < k < 1$, When $x = \frac{1}{3}$,

$\left\|\left(\frac{u_1-(1-x)e^x}{u_0-(1-x)e^x}\right)\right\| \le k = 0.0230013 < 1$, thus,

$\|u_1 - u\| \le k\|u_0 - (1-x)e^x\| = k\|u_0 - u\|$,

Then, we have

$\|u_1 - u\| \le k\|u_0 - u\|$,

$\|u_2 - u\| = \|u_2 - (1-x)e^x\| \le \|(u_1 - (1 -$
xex)(u2−1−xex u1−1−xex),

Since, $\forall x \in [0,1]$, $0 < k < 1$, When $x = \frac{1}{3}$,

$\left\|\left(\frac{u_2-(1-x)e^x}{u_1-(1-x)e^x}\right)\right\| \le k = 0.0236512 < 1$,

$\|u_2 - u\| \le k\|u_1 - (1-x)e^x\| = k\|u_1 - u\| \le kk\|u_0 - u\|$,

Thus, we get

$\|u_2 - u\| \le k^2\|u_0 - u\|$,

Similarly,

$\|u_3 - u\| = \|u_3 - (1-x)e^x\| \le \|(u_2 - (1 -$
xex)(u3−1−xex u2−1−xex),

Since, $\forall x \in [0,1]$, $0 < k < 1$, When $x = \frac{1}{3}$,

$\left\|\left(\frac{u_3-(1-x)e^x}{u_2-(1-x)e^x}\right)\right\| \le k = 0.0236351 < 1$,

$\|u_3 - u\| \le k\|u_2 - (1-x)e^x\| = k\|u_2 - u\| \le kk^2\|u_0 - u\|$,

$\|u_3 - u\| \le k^3\|u_0 - u\|$,

By continuing in this way we get:

$\|u_n - u\| \le k^n\|u_0 - u\|$,

Therefore,

$\lim_{n\to\infty}\|u_n - u\| \le \lim_{n\to\infty} k^n\|u_0 - u\| = 0$,

$\lim_{n\to\infty} k^n = 0$, then

$\lim_{n\to\infty}\|u_n - u\| = 0 \to \lim_{n\to\infty} u_n = u$, that is $u(x) = \lim_{n\to\infty} u_n(x) = (1-x)e^x$, which is an exact solution.

5. Conclusion:

The main objective of this paper has been achieved by solving initial and boundary value problems that appeared in engineering and applied science. The exact solutions for Riccati and Pantograph equations are obtained. However, we achieved the approximate solutions for beam equation through calculate the absolute error functions and better accuracy is obtained in comparison to HPM and VIM. The efficiency and accuracy of the TAM has been proved by studying the convergence and error analysis. It is seems that the TAM appears to be accurate to employ with reliable results and does not required any restricted assumption to deal with nonlinear terms.

References

[1] Abbasbandy. S., 2006, "Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method," Applied Mathematics and Computation, **172** (1), pp.485–490.

[2] Adomian, G., and Rach, R., 1986, "Solving nonlinear differential equations with decimal power nonlinearities," Journal of Mathematical Analysis and Applications, **114** (2), pp.423–425.

[3] Ahmadian, M., Mojahedi, M., and Moeenfard, H., 2009, "Free vibration analysis of a nonlinear beam using homotopy and modified lindstedt-poincare methods," Journal of Solid Mechanics, **1** (1), pp.29–36.

[4] AL-Jawary, M. A., and Al-Razaq, S. G., 2016, "A semi analytical iterative technique for solving Duffing equations," International Journal of Pure and Applied Mathematics, 108 (4), pp. 871–885.

[5] AL-Jawary, M. A., and Raham, R. K., 2016, "A semi-analytical iterative technique for solving chemistry problems," Journal of King Saud University, (In press).

[6] AL-Jawary, M. A., 2017, "A semi-analytical iterative method for solving nonlinear thin film flow problems," Chaos, Solitons and Fractals, **99**, pp.52–56.

[7] Aminikhah, H., 2013, "Approximate analytical solution for quadratic Riccati differential equation," Iranian Journal of Numerical Analysis and Optimization, **3** (2), pp.21–31.

[8] Anderson, B. D., and Moore, J. B., 1999, "Optimal control-linear quadratic methods," Prentice-Hall, New Jersey.

[9] Banach, S., 1922, "Sur les opérations dans les ensembles abstraits et leur application aux equations integrals," Fundamenta Mathematicae, **3**, pp. 133–181.

[10] Barari, A., Ganjavi, B., Jeloudar, M. G., and Domairry, G., 2010, "Assessment of two analytical methods in solving the linear and nonlinear elastic beam deformation problems," Journal of Engineering, Design and Technology, **8**, (2), pp.127–145.

[11] Barari, A., Omidvar, M., Ganji, D. D., and Tahmasebi, A. P., 2008, "An Approximate Solution for Boundary Value Problems in Structural Engineering and Fluid Mechanics," Journal of Mathematical Problems in Engineering, Vol. 2008, pp.1–13.

[12] Bellman, R. E., and Kalaba, R. E., 1965, "Quasilinearisation and nonlinear boundary-value problems," Elsevier, New York, **3**.

[13] Biazar, J., and Ghazvini, H., 2009, "Convergence of the homotopy perturbation method for partial differential equations," Nonlinear Analysis Real World Applications, **10** (5), pp. 2633–2640.

[14] Buhmann, M.D., and Iserles, A., 1993, "Stability the discretized pantograph differential equation," Mathematics of Computation, **60**, pp. 575–589.

[15] Cang, J., Tan, Y., Xu, H., and Liao, S. J., 2009, "Series solutions of non-linear Riccati differential equations with fractional order," Chaos,Solitons and Fractals, **40** (1), pp.1–9.

[16] Chawla, M. M., and Katti, C.P., 1979, "Finite difference methods for two-point boundary-value problems involving higher order differential equations," BIT Numerical Mathematics, **19**, (1), pp.27–33.

[17] Chowdhury, M. S. H., Hashim, I., and Abdulaziz, O., 2009, "Comparison of homotopy analysis method and homotopy-perturbation method for purely nonlinear fin-type problems," Communications in Nonlinear Science and Numerical Simulation, **14**, (2), pp.371–378.

[18] Doha, E. H., Bhrawy, A. H., Baleanu, D., and Hafez, R.M., 2014, "A new Jacobi rational–Gauss collocation method for numerical solution of generalized pantograph equations," Applied Numerical Mathematics, **77**, pp.43–54.

[19] Ehsani, F., Hadi A., Ehsani, F., and Mahdavi, R., 2013, "An iterative method for solving partial differential equations and solution of Korteweg-de Vries equations for showing the capability of the iterative method," World Applied Programming, **3** (8), pp.320–327.

[20] Fox, L., Mayers, D.F., Ockendon, J.A., and Tayler, A.B., 1971, "On a functional differential equation," IMA Journal of Applied Mathematics, **8** (3), pp.271–307.

[21] Genga, F., Lin, Y., and Cui M., 2009, "A piecewise variational iteration method for Riccati differential equations," Computers and Mathematics with Applications, **58** (11–12), pp.2518–2522.

[22] Ghomanjani, F., and Khorrama, E., 2017, "Approximate solution for quadratic Riccati differential equation," Journal of Taibah University for Science, **11** (2), pp.246–250.

[23] Glsu, M., and Sezer, M., 2006, "On the solution of the Riccati equation by the Taylor matrix method," Applied Mathematics and Computation, **176** (2), pp.414–421.

[24] He, J. H., 2000, "Variational iteration method for autonomous ordinary differential systems,"Applied Mathematics and Computation, **114** (2–3), pp.115–123.

[25] Khan, N. A., Ara, A., and Khan, N. A., 2013, "Fractional order Riccati differential equation analytical approximation and numerical results," Advances in Difference Equations, Accepted article, **185** (1), pp.1–16.

[26] Lasiecka, I., and Tuffaha, A., 2007, "Riccati equations arising in boundary control of fluid structure interactions," International Journal of Computing Science and Mathematics, **1** (1), pp.128–146.

[27] Li, Y., and Hu, L., 2010, "Solving fractional Riccati differential equations using Haar Wavelet," Third

International Conference on Information and Computing, **1**(5), pp. 314–317.
[28] Liao, S. J., and Chwang, A.T., 1998, "Application of homotopy analysis method in nonlinear oscillations," Journal of Applied Mechanics, **65** (4), pp.914–922.
[29] Ma, T. F., and Silva, J., 2004, "Iterative solution for a beam equation with nonlinear boundary conditions of third order," Applied Mathematics and Computation, 159 (1), pp.11–18.
[30] Momani, S. M., 1991, "Some Problems in Non-Newtonian Fluid Mechanics," PhD thesis,Wales University.
[31] Nayfeh, A. H., 1981, "Introduction to Perturbation Techniques," John Wiley Sons, New York, Chichester, pp.519.
[32] Odibat, Z., and Momani, S., 2008, "Modified homotopy perturbation method application to quadratic Riccati differential equation of fractional order," Chaos, Solitons and Fractals, 36 (1), pp.167–174.
[33] Saadatmandi, A., and Dehghanb, M., 2009, "Variational iteration method for solving a generalized pantograph equation," Computers and Mathematics with Applications, **58** (11–12), pp. 2190–2196.
[34] Tan, Y., and Abbasbandy, S., 2008, "Homotopy analysis method for quadratic Riccati differential equation," Communications in Nonlinear Science and Numerical Simulation, **13** (3), pp. 539–546.
[35] Temimi, H., and Ansare, A. R., 2011, "A new iterative technique for solving nonlinear second order multi-point boundary value problems," Applied Mathematics and Computation, **218** (4), pp.1457–1466.
[36] Tohidi, E., Bhrawy, A. H., and Erfani, K., 2013, "A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation," Applied Mathematical Modelling, 37 (6), pp.4283–4294.
[37] Wazwaz, A. M., 2014, "The variational iteration method for solving linear and nonlinear ODEs and scientific models with variable coefficients," Central European Journal of Engineering, **4** (1), pp.64–71.
[38] Yaman, M., 2007, "Adomian Decomposition Method for Solving a Cantilever Beam of Varying Orientation with Tip Mass," ASME Journal of Computational and Nonlinear Dynamics, **2**, pp.52–57.
[39] Yang, Y., and Huang, Y., 2013, "Spectral Collocation Methods for Fractional Pantograph Delay Integrodifferential Equations," Advances in Mathematical Physics, Vol. 2013, pp.14.
[40] Yusufoğlu, E., 2010, "An efficient algorithm for solving generalized pantograph equations with linear functional argument," Applied Mathematics and Computation, **217** (7), pp.3591–3595.
[41] Yuzbasi, S., and Sezer, M., 2013, "An exponential approximation for solutions of generalized pantograph-delay differential equations," Applied Mathematical Modeling, **37** (22), pp.9160–9173.