Approximation of Functions Belonging to the Lip $(\xi(t), p)$ and $W(L^p\xi(t))$ Class by Taylor - Cesaro Product Summability Method of its Fourier series

Yamini Yadav[1], C. S. Rathore[2]

[1]Department of Mathematics, Dr.C.V.Raman University Kota, Bilaspur (C.G.) India

[2]Department of Mathematics, Jajwalyadev Govt. Girls College, Janjgir (C.G.) India

Abstract: ***In this paper a new theorem established on the degree of approximation of function belonging to Lip $(\xi(t), p)$ and weighted $W(L^p\xi(t))$class by using Taylor-Cesaro product summability method of its fourier series.***

Keywords: Degree of approximation, Lip $(\xi(t), p)$ and weighted $W(L^p\xi(t))$ class, Taylor-Cesaro Summability method, Fourier series

1. Introduction

Bernstein (13), Alexits (5), Sahney & Goel (3), Chandra (11), Alexits & Leindler (5) and several others have determined the degree of approximation of a function $f \in Lip\alpha$ by (C, 1), (C, δ), (N, P_n) and ($\bar{N}$, P_n) means of its fourier series .Working in the same direction Sahney and Rao (9) and Khan (6) have studied the degree of approximation of function belonging to $Lip\alpha$ and $Lip(\alpha, p)$ by (N, P_n) means and (N, p, q) mean respectively.

Binod Prasad Dhakal (2) determined the degree of approximation of certain function belonging to the $Lip\ \alpha$ class by Matrix Cesaro summability method .Still today there is no work regarding the degree of approximation of function belonging to weighted $W(L^p\xi(t))$class by Taylor Cesaro summability method of its fourier series have done. Weighted $W(L^p\xi(t))$class is the generalization of $Lip\ \alpha, Lip(\alpha, p)$ and $Lip(\xi(t), p\ )$ classes.

Here we introduce a generalized theorem on the degree of approximation of function belonging to the weighted $W(L^p\xi(t))$class by Taylor Cesaro Summability method.

2. Definition and Notation

Let f be 2π periodic function, Lebesgue integrable and a function of Lip $(\boldsymbol{\alpha})$ and $W(L^p\xi(t))$ class over $(-\pi, \pi)$ in the sense of Lebesgue, then its fourier series is given by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \qquad (1)$$

We define the norm $\|\ \ \|_p$ by

$$\|f\|_p = \left\{\int_0^{2\pi} |f(x)|^p dx\right\}^{\frac{1}{p}}, p \geq 1$$

A function $f \in Lip\alpha$ if $|f(x+t) - f(x)| = O(|t|^{\alpha})$, for $0 < \alpha \leq 1$.

$$f \in Lip(\alpha, p)$$

if

$$\left\{\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right\}^{\frac{1}{p}} = O(|t|^{\alpha}), 0 < \alpha \leq 1. p \geq 1 .$$

Given a positive increasing function $\xi(t), p \geq 1$,

$f \in Lip(\xi(t), p)$ if $\left\{\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right\}^{\frac{1}{p}} = O(\xi(t)$

And $f \in W(L^p\xi(t))$ if

$$\left\{\int_0^{2\pi} \left|\{f(x+t) - f(x)\}Sin^{\beta} x\right|^p dx\right\}^{\frac{1}{p}} = O(\xi(t), \beta \geq 0 .$$

Let $\sum_{n=0}^{\infty} u_n$ be the infinite series whose n^{th} partial sum is given by $S_n = \sum_{k=0}^{\infty} u_k$.

Cesaro means (C, 1) of sequence $\{S_n\}$ is given by
$\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k$ .If $\sigma_n \to S$, as $n \to \infty$ then sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Cesaro means (C, 1) to S. It is denoted by
$\sigma_n \to S\big((\mathrm{C}, 1)\big)$, as $\to \infty$.

(2.1) The degree of approximation of a function f:R→ $R$by a trigonometric polynomial T_n of degree n is given by
$\| T_n - f\| = sup\{|T_n(x) - f(x)|: x\epsilon p\}$

(2.2) Let $\sum u_n$ be a given infinite series with sequence of its n^{th} partial sum $\{S_n\}$.The (C, 2) transform is defined as the n^{th} partial sum of (C, 2) summability and is given by

$$\sigma_n = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)S_k \to S \text{ as } n \to \infty$$

Then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by (C, 2) method.

(2.3) A given sequence $\{S_n\}$ is said to be Taylor summable, if $(T_n) = \sum_{k=0}^{n} u_{n,k}\ S_k \to S$ as $n \to \infty$, then the (C, 2)transform of Taylor means defines the (T_n, C_2) transform of the partial sum $\{S_n\}$of the series (2.1).
Thus, if $(T_n, C_2) = \sum_{k=0}^{n} u_{n,n-k}, \sigma_{n-k} \to S$ as $n \to \infty$
then $\sum_{n=0}^{\infty} u_n$ is said to be T_n, C_2 summable to S.

Remark: - We shall use following notations:
(i) $\emptyset(t) = f(x+t) - f(x-t) - 2f(x)$
(ii) $D(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n} \frac{u_{n,n-k}}{(n-k+2)} \frac{sin^2(n-k+2)t/2}{sin^2 t/2}$

3. Known Theorem

S.K.Tiwari and Vinita Sharma (12) have studied the degree of approximation of function belonging to Lip (α) class by using Taylor Cesaro product summability method of its fourier series.They proved the following theorem:

Theorem (3.1): If $f: R \to R$ is 2π periodic and lebesgue integrable on $[-\pi, \pi]$ and $f \in Lip\alpha$, then the degree of approximation of function by Taylor –Cesaro product means of the series, satisfies for n=0, 1, 2…,

$$\|T_n C_2(x) - f(x)\|_\infty = \begin{cases} O\left(\frac{1}{(n+2)^\alpha}\right); 0 < \alpha < 1 \\ O\left(\frac{\log(n+2)\pi e}{n+2}\right); \alpha = 1 \end{cases}$$

Where $T_n = a_{n,k}$ is non negative, monotonic and non-increasing sequence of real constant such that $\left|\sum_{k=0}^{n} u_{n,n-k}\right| = O(1)$.

4. Main Theorem

Theorem (4.1): If $f: R \to R$ is 2π periodic and lebesgue integrable on $[-\pi, \pi]$ and $f \in Lip(\xi(t), p)$, then the degree of approximation of function by Taylor –Cesaro product means of the Fourier series (1) is given by

$$\|T_n C_2(x) - f(x)\|_p = O\left((n+2)^{\frac{1}{p}} \xi\left(\frac{1}{n+2}\right)\right)$$

Provided $\left\{\frac{\xi(t)}{t}\right\}$is monotonic decreasing and $\xi(t)$ satisfy the following conditions:

$$\left\{\int_0^{\frac{1}{n+2}} \left(\frac{t|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} = O\left(\frac{1}{n+2}\right) \quad (4)$$

$$\left\{\int_{\frac{1}{n+2}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} = O\{(n+2)^\delta\} \quad (5)$$

Theorem (4.2): If $f: R \to R$ is 2π periodic and lebesgue integrable on $[-\pi, \pi]$ and $f \in W(L^p \xi(t))$ then the degree of approximation of function by Taylor –Cesaro product means of the Fourier series(1) is given by $\|T_n C_2(x) - f(x)\|_p = O\left\{(n+2)^{\beta + \frac{1}{p}} \xi\left(\frac{1}{n+2}\right)\right\}$

Provided $\xi(t)$ satisfies the following conditions: -

$$\left\{\int_0^{\frac{1}{n+2}} \left(\frac{t|\phi(t)|}{\xi(t)}\right)^p \sin^{\beta p} t dt\right\}^{\frac{1}{p}} = O\left(\frac{1}{n+2}\right) \quad (6)$$

$$\left\{\int_{\frac{1}{n+2}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} = O\{(n+2)^\delta\} \quad (7)$$

5. Required Lemmas

Lemma 5.1: For $0 \le t \le \frac{1}{n+2}$; $D(n,t) = O(n+2)$.

Proof: We have

$|D(n,t)| = \left|\frac{1}{2\pi} \sum_{k=0}^{n} \frac{u_{n,n-k}}{(n-k+2)} \frac{\sin^2 (n-k+2)t/2}{\sin^2 t/2}\right|$

$\le \frac{1}{2\pi} \left|\frac{u_{n,n-k}}{(n-k+2)} \frac{(n-k+2)^2 t^2/\pi^2}{t^2/\pi^2}\right|$

$= O(n+2) \left|\sum_{k=0}^{n} u_{n,n-k}\right|$

$= O(n+2)$

Lemma 5.2: For $\frac{1}{(n+2)} \le t \le \pi$; $D(n,t) = O\left(\frac{1}{(n+2)t^2}\right)$

Proof: We Have

$|D(n,t)| = \left|\frac{1}{2\pi} \sum_{k=0}^{n} \frac{u_{n,n-k}}{(n-k+2)} \frac{\sin^2 (n-k+2)t/2}{\sin^2 t/2}\right|$

Using Jordan's lemma $\sin\frac{t}{2} \ge \frac{t}{\pi}$ and $\sin kt \le 1$; we have

$\le \frac{1}{2\pi} \left|\frac{u_{n,n-k}}{(n-k+2)} \frac{1}{t^2/\pi^2}\right|$

$= O\left(\frac{1}{n+2}\right) \left|\sum_{k=0}^{n} u_{n,n-k}\right|$

$= O\left(\frac{1}{(n+2)t^2}\right)$

6. Proof of the Main Theorem

Proof of Theorem 4.1:

Let $S_n(x)$denote the n[th] partial sum of the series (2.1) at $t = x$, then the following Titchmarch [5],

we have

$$\sigma_n(x) - f(x) = \frac{2(n-k+1)}{2\pi(n+1)(n+2)} \int_0^{\pi} \frac{\sin^2 (n+2)t/2}{\sin^2 t/2} dt$$

Now, the Taylor, transform of the sequence $\{\sigma_n\}$ is given by

$\sum_{k=0}^{n} u_{n,n-k} \{\sigma_n(x) - f(x)\} =$

$\frac{2}{2\pi} \int_0^{\pi} \emptyset(t) \frac{u_{n,n-k}}{(n-k+2)} \frac{\sin^2 (n+2)t/2}{\sin^2 t/2} dt$;at k=0.

or $C_2(x) - f(x) = 2 \int_0^{\pi} \emptyset(t) D(n,t) dt$

$= 2\left[\int_0^{\frac{1}{n+2}} \emptyset(t) D(n,t) dt + \int_{\frac{1}{n+2}}^{\pi} \emptyset(t) D(n,t) dt\right]$

$= 2[I_{1.1} + I_{1.2}]$, Say (8)

Let us consider $I_{1.1}$ first

$$|I_{1.1}| = \left|\int_0^{\frac{1}{n+2}} \emptyset(t) D(n,t) dt\right|$$

$\le \int_0^{\frac{1}{n+2}} |\emptyset(t)||D(n,t)| dt$

Appling Holder's inequality and fact that $\emptyset(t) \in Lip(\xi(t), p)$, we have

$|I_{1.1}|$

$$\le \left\{\int_0^{\frac{1}{n+2}} \left(\frac{t\,|\emptyset(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} \cdot \left\{\int_0^{\frac{1}{n+2}} \left(\frac{\xi(t)|D(n,t)|}{t}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= O\left(\frac{1}{n+2}\right) O(n+2) \left\{\int_0^{\frac{1}{n+2}} \left(\frac{\xi(t)}{t}\right)^q dt\right\}^{\frac{1}{q}}$$

By condition (4) and Lemma I

Mean value theorem for integrals.

$$= O\left(\xi\left(\frac{1}{n+2}\right)\right) \left[\left(\frac{t^{-q+1}}{-q+1}\right)_0^{\frac{1}{n+2}}\right]^{\frac{1}{q}}$$

$$= O\left((n+2)^{1-\frac{1}{q}} \xi\left(\frac{1}{n+2}\right)\right)$$

$$I_{1.1} = \left((n+2)^{\frac{1}{p}} \xi\left(\frac{1}{n+2}\right)\right) \because \left[\frac{1}{p} + \frac{1}{q} = 1\right] \quad (9)$$

Let us consider $I_{1.2}$.

Apply Holder inequality and taking δ as an arbitrary number, we have

$|I_{1.2.}|$

$$\leq \left\{\int_{\frac{1}{n+2}}^{\pi}\left(\frac{t^{-\delta}\,|\emptyset(t)|}{\xi(t)}\right)^{p} dt\right\}^{\frac{1}{p}} \cdot \left\{\int_{\frac{1}{n+2}}^{\pi}\left(\frac{\xi(t)|D(n,t)|}{t^{-\delta}}\right)^{q} dt\right\}^{\frac{1}{q}}$$

$$= O\left((n+2)^{\delta}\right) O\left(\frac{1}{(n+2)}\right)\left\{\int_{\frac{1}{n+2}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+2}}\right)^{q} dt\right\}^{\frac{1}{q}}$$

By condition (5) and Lemma II

$$= O\left((n+2)^{\delta-1}\right)\left\{\int_{n+2}^{\pi}\left(\frac{\xi(t)}{t^{2-\delta}}\right)^{q} dt\right\}^{\frac{1}{q}}$$

$$= O(n+2)^{\delta-1}\left\{\int_{n+2}^{\pi}\left(\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-2}}\right)^{q}\frac{dy}{y^{2}}\right\}^{\frac{1}{q}} \text{ [taking } t=\frac{1}{y}]$$

$$= O(n+2)^{\delta-1}\xi\left(\frac{1}{n+2}\right)\left\{\int_{n+2}^{\pi} y^{-q(\delta-2)-2} dy\right\}^{\frac{1}{q}}$$

By mean value theorem for integrals

$$= O\left\{(n+2)^{\delta-1}\xi\left(\frac{1}{n+2}\right)\right\}\left[\left\{y^{-q(\delta-2)-1}\right\}^{\frac{1}{q}}\right]_{n+2}^{\pi}$$

$$= O\left\{O(n+2)^{\delta-1}\xi\left(\frac{1}{n+2}\right)\right\}\left[y^{-(\delta-2)-\frac{1}{q}}\right]_{n+2}^{\pi}$$

$$= O\left\{(n+2)^{\delta-1}\xi\left(\frac{1}{n+2}\right)\right\}(n+2)^{-\delta+2-\frac{1}{q}}$$

$$= O\left\{\xi\left(\frac{1}{n+2}\right).(n+2)^{1-\frac{1}{q}}\right\}$$

$$I_{2.2} = O\left\{(n+2)^{\frac{1}{p}}\xi\left(\frac{1}{n+2}\right)\right\} \because \left[\frac{1}{p}+\frac{1}{q}=1\right] \qquad (10)$$

From(8), (9) and (10) we have

$$\|T_n C_2(x)-f(x)\|_p = 2.O\left\{(n+2)^{\frac{1}{p}}\xi\left(\frac{1}{n+2}\right)\right\}$$

Theorem 4.2

$$C_2(x)-f(x) = 2\int_0^{\pi}\emptyset(t)\,D(n,t)dt$$

$$= 2\left[\int_0^{\frac{1}{n+2}}\emptyset(t)D(n,t)dt + \int_{\frac{1}{n+2}}^{\pi}\emptyset(t)D(n,t)dt\right]$$

$= 2[I_{2.1}+I_{2.2}]$, Say (11)

Let us consider $I_{2.1}$ first

$$|I_{2.1}| = \left|\int_0^{\frac{1}{n+2}}\emptyset(t)D(n,t)dt\right|$$

$$\leq \int_0^{\frac{1}{n+2}}|\emptyset(t)||D(n,t)|dt$$

Appling Holder's inequality and fact that $\emptyset(t)\in W(L^p\xi(t))$, we have

$|I_{2.1}|$

$$\leq \left\{\int_0^{\frac{1}{n+2}}\left(\frac{t\,|\emptyset(t)|sin^{\beta}t}{\xi(t)}\right)^{p} dt\right\}^{\frac{1}{p}} \cdot \left\{\int_0^{\frac{1}{n+2}}\left(\frac{\xi(t)|D(n,t)|}{t sin^{\beta}t}\right)^{q} dt\right\}^{\frac{1}{q}}$$

$$= O\left(\frac{1}{n+2}\right)O(n+2)\left\{\int_0^{\frac{1}{n+2}}\left(\frac{\xi(t)}{t^{1+\beta}}\right)^{q} dt\right\}^{\frac{1}{q}},$$

By condition (4) and Lemma I

$$= O\left(\xi\left(\frac{1}{n+2}\right)\right)\left\{\int_0^{\frac{1}{n+2}} t^{-(1+\beta)q}\,dt\right\}^{\frac{1}{q}}$$

Mean value theorem for integrals.

$$= O\left(\xi\left(\frac{1}{n+2}\right)\right)\left[\left(t^{-(1+\beta)q+1}\right)^{\frac{1}{q}}\right]_0^{\frac{1}{n+2}}$$

$$= O\left(\xi\left(\frac{1}{n+2}\right)\right)\left[t^{-(1+\beta)+\frac{1}{q}}\right]_0^{\frac{1}{n+2}}$$

$$= O\left(\xi\left(\frac{1}{n+2}\right)\right)(n+2)^{(\beta+1)-\frac{1}{q}}.$$

$$I_{2.1} = O\left(\xi\left(\frac{1}{n+2}\right)\right)(n+2)^{\beta+\frac{1}{p}} \because \left[\frac{1}{p}+\frac{1}{q}=1\right] \qquad (12)$$

Let us consider $I_{2.2}$.

$$|I_{2.1}| = \left|\int_0^{\frac{1}{n+2}}\emptyset(t)D(n,t)dt\right|$$

$$\leq \int_0^{\frac{1}{n+2}}|\emptyset(t)||D(n,t)|dt$$

Apply Holder inequality and taking δ as an arbitrary number, we have

$|I_{2.2.}|$

$$\leq \left\{\int_{\frac{1}{n+2}}^{\pi}\left(\frac{t^{-\delta}\,|\emptyset(t)|sin^{\beta}t}{\xi(t)}\right)^{p} dt\right\}^{\frac{1}{p}} \cdot \left\{\int_{\frac{1}{n+2}}^{\pi}\left(\frac{\xi(t)|D(n,t)|}{t^{-\delta}sin^{\beta}t}\right)^{q} dt\right\}^{\frac{1}{q}}$$

$$= O\left((n+2)^{\delta}\right) O\left(\frac{1}{(n+2)}\right)\left\{\int_{\frac{1}{n+2}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+\beta+2}}\right)^{q} dt\right\}^{\frac{1}{q}}$$

By condition (5) and Lemma II

$$= O\left((n+2)^{\delta-1}\right)\left\{\int_{n+2}^{\pi}\left(\frac{\xi(t)}{t^{2-\delta}}\right)^{q} dt\right\}^{\frac{1}{q}}$$

$$= O(n+2)^{\delta-1}\left\{\int_{n+2}^{\pi}\left(\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-\beta-2}}\right)^{q}\frac{dy}{y^{2}}\right\}^{\frac{1}{q}} \text{ [taking } t=\frac{1}{y}]$$

$$= O(n+2)^{\delta-1}\xi\left(\frac{1}{n+2}\right)\left\{\int_{n+2}^{\pi} y^{-q(\delta-\beta-2)-2} dy\right\}^{\frac{1}{q}}$$

By mean value theorem for integrals

$$= O\left\{(n+2)^{\delta-1}\xi\left(\frac{1}{n+2}\right)\right\}\left[\left\{y^{-q(\delta-\beta-2)-1}\right\}^{\frac{1}{q}}\right]_{n+2}^{\pi}$$

$$= O\left\{O(n+2)^{\delta-1}\xi\left(\frac{1}{n+2}\right)\right\}\left[y^{-(\delta-\beta-2)-\frac{1}{q}}\right]_{n+2}^{\pi}$$

$$= O\left\{(n+2)^{\delta-1}\xi\left(\frac{1}{n+2}\right)\right\}(n+2)^{-\delta+\beta+2-\frac{1}{q}}$$

$$= O\left\{\xi\left(\frac{1}{n+2}\right).(n+2)^{\beta+1-\frac{1}{q}}\right\}$$

$$I_{2.2} = O\left\{(n+2)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+2}\right)\right\} \because \left[\frac{1}{p}+\frac{1}{q}=1\right] \qquad (13)$$

From(11), (12) and (13) we have

$$\|T_n C_2(x)-f(x)\|_p = 2.O\left\{(n+2)^{\beta+\frac{1}{p}}\xi\left(\frac{1}{n+2}\right)\right\}$$

Thus the theorem is completely proved

References

[1] A.Zygmund: Trignometric series, Cambridge University Press, 1959.

[2] Binod Prsad Dhakal: Approximation of function f of Lip $(\xi(t),p)$ class by (C, 1), (N, p_n) method of its fourier series, Kathmandu Uni.J.of Sc., Eng.and Tech., Vol.9, No.I, July (2013), pp 145-151.

[3] B.N.Sahney and D.S. Goel: On degree of approximation of continuous functions, RanchiUni.Math. J.4(1973), 50-53.
[4] E.G.Titchmarsh: The theory of functions, Second Edition, Oxford University Press, (1939).
[5] G.Alexits: $\ddot{U}$ber die Ann$\ddot{a}$herung einer stetigen function durch die Cesaroschen Mittel in hrer Fourier reihe, Math .Annal 100 (1928), 264-277.
[6] Huzoor H. Khan: On the degree of approximation of function belonging to the class $Lip(\alpha, p)$ Indian J.Pure and Appl.Math.5(1974), 132-136.
[7] K.Qureshi & H.K.Nema:A class of function and their degree of approximation, Ganita, 41(1-2)(1990), 37.
[8] K.Qureshi & H.K.Nema: On the degree of approximation of functions belonging to the weighted class, Ganita, 41 (1-2) (1990), 17.
[9] N.Sahney and Gopal Rao: Errors bound in the approximation function, Bull.Aust. Math.Soc.6(1972), 11- 18.
[10] O.T$\ddot{o}$eplitz: $\ddot{U}$berall gemeine lineare Mittelbildungen, Prace mat.-fiz., 22(1913), 113-119.
[11] PremChandra: On the degree of approximation of functions belonging to the Lipschitz class, Nanta Math ., 80(1975), 88-89.
[12] S.K.Tiwari and Vinita Sharma: A study on the Taylor – Cesaro Product summability method of Fourier series, International Research Journal of pure Algebra-4(12), 2014, 653-657.
[13] S.N.Bernstein: Sur 1' order de la meilleure approximation des functions continues par les polynomes de degree donne'e, Memories Acad.Roy-Belyique, 4(1912), 1-104.