FI-Semihollow and FI- Semilifting Modules

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Abstract: Let R be a commutative ring with identity and let M be a left unital R-module. A submodule N of R-module M is called small in M (denoted by \(N \subseteq M\)) if \(N + K \neq M\) for each proper submodule \(K\) of \(M\). A submodule \(N\) of R-module M is called a hollow module if every proper submodule is small in \(M\). A submodule \(N\) of R-module M is called fully invariant submodule of M if \(f(N) \subseteq N\), for every \(f \in \text{Hom}(M,M)\).

Keywords: semihollow module, semilifting module, fully invariant submodule.

1. Introduction

Let \(R\) be a commutative ring with identity and \(M\) be a left unital R-module. A submodule \(N\) of R-module \(M\) is called small in \(M\) (denoted by \(N \subseteq M\)) if \(N + K \neq M\) for each proper submodule \(K\) of \(M\). A submodule \(N\) of R-module \(M\) is called a hollow module if every proper submodule is small in \(M\). A submodule \(N\) of R-module \(M\) is called fully invariant submodule of \(M\) if \(f(N) \subseteq N\), for every \(f \in \text{Hom}(M,M)\).

In [4] there was given the concepts of semismall submodule semihollow modules and semilifting modules as a generalization of the concepts of small submodule, hollow modules and lifting modules. Where a submodule \(N\) of R-module \(M\) is called semismall (\(N \subseteq M\)) if \(N = 0\) or for each nonzero submodule \(K\) of \(M\), then \(N/K \subseteq M/K\).

In this paper we introduce the concept of FI-semihollow module and FI-semilifting module as a generalization of semihollow and semilifting modules and study the basic properties of this type of modules and give some characterizations for such modules.

Lemma (1):[4]

Let \(N\) be a submodule of R-module \(M\), \(N \subseteq M\) iff \(N + L = M\) for all \(L \subseteq M\) implies \(K + L = M\) for all \(K \subseteq N\), \(K \neq (0)\).

Lemma (2):[4]
1) If \(N \subseteq M\), \(M\) and \(A < N\) then \(A \subseteq M\).
2) If \(X, Y\) are submodule of \(M\) such that \(X \subseteq Y\), then \(X \subseteq M\).
3) If \(N \subseteq M\) and \(K < M\) such that \(K \subseteq N\) then \(N/K \subseteq M/K\).
4) If \(M = M_1 \oplus M_2\) and \(N \subseteq M\) such that \(N = N_1 \oplus N_2\), if \(N \subseteq M\), then \(N_1 \subseteq N_2\) and \(N_2 \subseteq N_1\).
5) If \(N \not\subseteq M\) and \(N \not\subseteq M\) then \(N \not\subseteq N\).

1- FI-Semi Hollow module

An non-zero R-module \(M\) is called semihollow if every proper submodule of \(M\) is Semismall, [4]. In this section we introduce the concept of FI-semihollow module and we study the properties of this concept.

Definition (1.1):- An non zero R-module \(M\) is called FI-semihollow if every fully invariant proper submodule of \(M\) is Semismall.

A ring \(R\) is called FI-semihollow if \(R\) is FI-semihollow as an R-module, Equivalently every two sided ideal of \(R\) is semismall.

Remarks and Examples (1.2):-
1) Every hollow and semihollow is FI-semihollow module.
2) IF \(M\) is Duo module then semihollow and FI-semihollow are equivalent.
3) \(Z\) as Z-module is not semihollow and not FI-semihollow since \(2Z\) is fully invariant proper submodule and \(2Z + 2Z = Z\) but \(6Z + 3Z = M, 6Z < 2Z\).
4) \(Z \oplus Z\) as Z-module is FI-semihollow which is not Duo module.
5) Every simple R-module is FI-semihollow.

Proposition (1.3):

FI-semihollow is closed under isomorphism.

Proof:
Let \(M, M'\) be R-modules and \(M\) is FI-semihollow, let \(f: M \rightarrow M'\) be an R-isomorphism, we have to show that \(M'\) is FI-semihollow, let \(N\) be fully invariant proper Submodule of \(M'\), \(N = f^{-1}(N)\) is proper submodule of \(M\). If \(f^{-1}(N) = M\) then \(f(f^{-1}(N)) = N = M\) which is contradiction. Thus \(f^{-1}(N)\) is proper submodule if \(M\) to show \(f^{-1}(N)\) is fully invariant of \(M\). Let \(g: M \rightarrow M\), since \(N\) is fully invariant of \(M\), thus \(f^{-1}(N) \subseteq N\) and \(f^{-1}(N) \subseteq f^{-1}(N)\), then \(g(f^{-1}(N)) \subseteq f^{-1}(N)\). Thus \(f^{-1}(N)\) is fully invariant of \(M\), since \(M\) is FI-semihollow , thus \(f^{-1}(N) \subseteq M\) and by [4] prop. (1.3) and \(N \subseteq M'\).

Proposition (1.4):

Let \(M\) be a FI- semihollow and \(N\) be a submodule of \(M\) with \(N/K\) is direct summand of \(M/K\) for each proper submodule \(K\) of \(N\) then \(N\) is FI-semihollow.

Proof: Let \(L\) be a proper fully invariant submodule of \(M\) then \(L \subseteq M\), by Lemma(2).
Let \(K\) be a proper submodule of \(L\) then \(L/K \subseteq M/K\) and by hypothesis \(L/K\) is a direct summand of \(M/K\) then \(L/K \subseteq N/M\) hence \(L \subseteq M\).
Remark (1.5): FI-semihollow need not be indecomposable, $Z_e$ as $Z$-module is semihollow then FI-semihollow which is decomposable.

Recall that a submodule $N$ of an $R$-module $M$ is called coclosed in $M$ if whenever $N/K \leq M/K$ then $N=K \cap K$ submodule of $M$ contained in $N$.

This means $N$ is coclosed if whenever $K<N$, $N/K$ is not small in $M/K$, it is known that the only proper coclosed submodule in semihollow module is simple submodule.

We have the following Remark.

Remark: Every proper fully invariant coclosed submodule of a semihollow module is a simple submodule.

Proof: Let $N$ be fully proper coclosed submodule of FI-semihollow module $M$, since $M$ is FI-semihollow then $N/K \leq M/K \cap K \leq N$ since $N$ is coclosed, $N=K$ then $N$ issimple.

Corollary: Every non-zero coclosed fully invariant submodule of FI-semihollow module is FI-semihollow.

Proof: Since every simple is FI-semihollow.

2. FI-Semilifting modules

An $R$-module $M$ is called semilifting if for any submodule $N$ of $M$ there exist submodules $K$, $K'$ of $M$ such that $M=K \oplus K'$ with $K \subseteq K$ and $N \cap K \leq N$ (equivalently $N \cap K' \leq N$). In this section we introduce the notion of FI-semilifting modules and discuss some properties of this kind of modules which are generalization of semilifting module.

Definition (2.1) :
An $R$-module $M$ is called FI-semilifting if for any fully invariant submodule $N$ of $M$ there exist submodules $K$, $K'$ of $M$ such that $M=K \oplus K'$ with $K \subseteq N$ and $N \cap K' \leq N$. [4] The following theorem given a characterization of FI-semilifting modules.

Theorem (2.2) : Let $M$ be an $R$-module, then the following statement are equivalent :
1) $M$ is FI-semilifting
2) Every fully invariant submodule $N$ of $M$, $N$ can be written as $N=A \oplus B$ where $A$ is direct summand of $M$ and $B \subseteq M$.
3) For every fully invariant submodule $N$ of $M$, there exist a direct summand $K$ of $M$ such that $K \leq N$ and $N/K \leq N$.

Proof: It is clear same like theorem 3.3 in [4]. It is known that every hollow is lifting, but the converse is not true (see Remark 1.1.7. in 11).

Remark (2.3) :
1) Every FI-semihollow is FI-semilifting
Proof: Let $N$ be fully invariant submodule of $M$ if $N\neq M$, $N\leq M$, $N=(0) \oplus N$ the result follows directly by (Theorem 2.2).
2) Every semisimple module is lifting hence semilifting and then FI-semilifting.

Proposition (2.4) :
An indecomposable $R$-module is FI-semihollow if and only if FI-semilifting.

Proof: Let $M$ be FI-semihollow then $M$ is FI-semilifting by Remark (2.3)

Conversely suppose that $M$ is FI-semilifting and $A$ fully invariant proper submodule of $M$, by (Theorem 2.2), we have $A=N \oplus D$ where $N$ is a direct summand of $M$ and $D \leq N$ but $M$ is indecomposable. Then either $N=(0)$ or $N=M$ then $M=N \subseteq A$ which implies that $A=M$ which is contradiction, thus $N=(0)$, So $A=D \leq M$, hence $M$ is FI-semihollow.

Proposition (2.6) :
If an $R$-module $M$ is FI-semihollow then $M/N$ is FI-semihollow for every fully invariant proper submodule $N$ of $M$.

Proof: Assume that $M$ is FI-semihollow and let $N$ be fully invariant proper submodule of $M$

Let $K/N$ be fully invariant proper submodule of $M/N$, such that $M/N=K/N \oplus H/N$ where $H \subseteq M$ and $N \cap H \leq N$. Then $M=K+H$, now since $K/N$ is fully invariant of $M/N$ and $N$ is fully invariant proper of $M$ thus $K$ is fully invariant by (Lemma 1.2.23(6)) then $K \leq M$, So $K+H=M$, $M \leq K$ then $K'=H=K \cap K$ (Lemma 1). Hence we have $K+H=M/N$ then $M/K \cap K'=N/K \leq K' \neq N$, thus $M/N$ is FI-semihollow module.

References