Historical Developments in Fractional Calculus: A Survey

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Abstract: ***In today's world of science the study of fractional calculus become a trending field of research. Since fractional calculus plays an important role in the various applications in science and engineering, the field has been grown so far.The aim of this paper is to give a detailed survey on historical developments in fractional calculus. In the present article we survey on historical developments in fractional calculus and most popular definitions of the fractional derivative and fractional integration.***

Keywords: Historical developments, Gamma function, Beta function Fractional integral, fractional derivative

1. Introduction

Fractional calculus does not mean the calculus of fraction or the fraction of derivative and integration. Fractional calculus means the derivative and integration of arbitrary order. In fact fractional calculus is the extension of integer order calculus.

Fractional calculus was born in the letters of L' Hospital to Leibnitz in 1695. Leibnitz used the notation $\frac{D^n f(x)}{Dx^n}$ for nth order derivative of a function $f(x)$ in one of his publication. L'Hospital wrote a letter to Leibnitz and asked him "what would be the result if $n = 1/2$". Leibnitz replied him "an apparent paradox from which one day useful consequences will be drawn". In these words fractional calculus was born. Liouville(1832) expanded the functions in series of exponentials and defined the qth derivative by operating term by term differentiation of series. Grunwald in 1867 derived the definite integral formula for qth derivative.Whereas Riemann in 1953 proposed a new definition that involved a definite integral. Further this formula was applicable to power series with fractional exponents. Grunwald and Krug unified the results of Liouville and Riemann. In these theoretical beginning the applications of fractional calculus to various problems was developed. The first application of fractional calculus was developed by Abel in 1823.He found that the solution of integral equation of a tautochrone can be obtained by using integral transforms. The symbolic methods for solving a linear equations with constant coefficient was developed by Boole in 1844. In 1892 Heaviside developed the operational calculus to find the solutions of certain problems in electromagnetic theory. Further in 1920 he introduced the concept of fractional differentiation in his investigation of transmission line theory. In 1936 Gemat extended this Heaviside's concept of fractional differentiation for the use of problems in elasticity. Riesz (1949) has developed the theory and applications of factional integration of function of more than one variable. In 1953 Kuttner investigated some natural properties of integration and differentiation of arbitrary order of functions belonging to Lebesgue and Lipschitz classes.

Recently the study of fractional calculus has become a most popular field of research in Mathematics, Physics and Engineering. In the present century tremendous work have been made to both theory and applications of fractional calculus. The mathematicians who have contributed directly and indirectly in the development of fractional calculus are Holmgren (1865-18670), A), H. Laurent (1884), P. A. Nekrassov (1888), A. Kurg (1890), J. Hadmard (1892), O. Heaviside (1892,1893,1920), Hardy and Littlewood (1925,1928,1932), H.Weyl (1917), Buss(1929), P.Levy (1923), A. Marchaud (1927), H. Devis (1924,1927), Post(1930), Kober (1940), Goldman (1949), Scott Blair (1949), Kuttner (1953), M.M. Dzherbashyan and A.B. Nersesian (1958,1999), Erdelyi (1964), Higgins (1967), Oldham and Spanier (1974), L. Debnath (1992), Miller Ross (1993), A. A. Kilbass (1993), R. Gorenflo and F. Mainardi (2000), I. Podlubny (2003), X. J. Yang (2012), and many more.

2. Historical Developments

In 1730 Euler obtained the derivative of fractional order.

$$\frac{d^n t^m}{dt^n} = m(m-1)(m-2)\ldots\ldots\ldots(m-(n-1))t^{m-n}$$

$$\therefore \frac{d^n t^m}{dt^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}$$

Putting $m = 1 \ and \ n = 1/2$ we get,

$$\frac{d^{\frac{1}{2}} t}{dt^{\frac{1}{2}}} = \sqrt{\frac{4t}{\pi}} = \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}$$

J. B. Fourier (1820) introduced the integral,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos(pt - pz)dp$$

Using this integral he obtained the definition of nth order derivative for non-integer order n as,

$$\frac{d^n}{dt^n} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \mathrm{p}^{\mathrm{n}} \cos\left(pt - pz + \frac{n\pi}{2}\right) dp$$

Fourier also mentioned that the number n appears in above definition will be positive or negative.

N. H. Abel (1823-1826) extended the definition for arbitrary number α. He introduced the integral as,

$$f(t) = \int_0^t \frac{g'(\chi)}{(t-\chi)^\alpha} d\chi$$

Abel solved the above integral for arbitrary α

$$g(t) = \frac{\sin(\pi\alpha)}{\pi} t^\alpha \int_0^1 \frac{f(tx)}{(1-x)^{1-\alpha}} dx$$

With the help of integral of order α, he obtained the solution as,

$$g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d^{-\alpha}}{dt^{-\alpha}} f(t)$$

Abel uses the fractional calculus to find the solution of tautochrone problem. Tautochrone problem is to find the shape of a curve where the time of descent is independent of the position of release of ball in a frictionless system, sliding down the curve under the action of gravity. In this case the Abel's equation is,

$$k = \int_0^t (t-x)^{-\frac{1}{2}} f(x) dx = \sqrt{\pi} \frac{d^{-\frac{1}{2}}}{dt^{-\frac{1}{2}}} f(t)$$

This equation is a particular case of definite fractional integral of order 1/2.Operating $d^{\frac{1}{2}} / dx^{\frac{1}{2}}$ on both sides of above equation gives,

$$\frac{d^{\frac{1}{2}}}{dt^{\frac{1}{2}}} k = \sqrt{\pi} f(t)$$

With the help of above definition, Abel conclude that the fractional derivative of constant need not be zero.

J. Liouville (1832-1855) obtained the definition of fractional derivative as,

$$\frac{d^\alpha f(t)}{dt^\alpha} = \lim_{h\to 0} \frac{(-1)^\alpha}{h^\alpha} \left(f(t) - \frac{\alpha}{1} f(t+h) + \frac{\alpha(\alpha-1)}{1.2} f(t+2h) - \cdots \ldots.. \right)$$

and,

$$\frac{d^\alpha f(t)}{dt^\alpha} = \lim_{h\to 0} \frac{1}{h^\alpha} \left(f(t) - \frac{\alpha}{1} f(t-h) + \frac{\alpha(\alpha-1)}{1.2} f(t-2h) - \cdots \ldots.. \right)$$

Riemann (1847) obtained the formula for integration of non-integer order using the Taylor series. Since Riemann did not fix the lower bound of integration, he introduced the complementary function $\psi(t)$

$${}_cD_t^{-\alpha} f(t) = {}_cI^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx + \psi(t),$$

From this definition the initialized fractional calculus was born in the middle half of twentieth century.

Marchaud (1927) defined the fractional derivative from Riemann-Liouville fractional integral by replacing α *by* $-\alpha$

$${}_0D_\infty^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty u^{-\alpha-1} f(t-u) du \quad , for\ \alpha > 0$$

As $u \to 0$, the above integral diverges. Therefore this definition was modified for $0 < \alpha < 1$ as,

$${}_{0^+}D_t^\alpha f(t) = \lim_{\epsilon\to 0^+} \frac{1}{\Gamma(1-\alpha)} \int_\epsilon^\infty u^{-\alpha} f'(t-u) du$$

Riesz (1949) defined the fractional integral as,

$$I_\alpha f(t) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \int_{-\infty}^\infty \frac{f(u)du}{|t-u|^{n-\alpha}}$$

B. Ross in 1974 organized a first conference on fractional calculus and its applications. He edited the proceedings of the conference [7]. K.B. Oldham and J. Spanier (1974) published a book on fractional calculus [8]. The detailed analysis physical applications of fractional calculus is listed in the work of M.M. Dzherbashyan [9, 10], M. Caputo[11], Gorenflo and Vessella [12], Samko, Kilbas and Marichev [13], Babenko[14] and many more in the titles of [11, 15, 16, 17, 18, 19, 20, 21, 22, 23]

Recently many of the mathematician has proved that the derivative and integration of arbitrary order are more convenient that the integer order in describing the properties of real materials. From last two decades the tremendous work is made in fractional calculus and its applications. Many of them have developed the iterative method and finite difference methods for solving fractional linear and nonlinear differential equations

3. Functions useful in Fractional Calculus

i) **The Gamma Function:**

The gamma function on a complex plane is defined as,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad , \qquad Re(z) > 0 \qquad (1)$$

The gamma function plays an important role in the fractional calculus.

We can easily verify that,

$$\Gamma(z+1) = z\Gamma(z)$$

In n is a positive integer then,

$$\Gamma(n+1) = n!$$

ii) The Beta function

The beta function defined on a complex plane is given by,

$$\mathrm{B}(z,w) = \int_0^1 t^{z-1} (1-t)^{w-1} dt \ , \qquad Re(z) > 0\ and\ Re(w) > 0$$

wher $Re(\alpha) > 0$

From the definition we observed that beta function is symmetric

$$i.e.\ \mathrm{B}(z,w) = \mathrm{B}(w,z)$$

The relation between the beta function and the gamma function given by

$$\mathrm{B}(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

iii) The Mittage-Leffler function

The Mittage-Leffler function is very useful in the fractional derivative and integration.It is the generalization of an exponential function.

The one parameter Mittage-Leffler function is given by,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha > 0$$

The two parameter Mittage-Leffler function is given by,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha > 0 \text{ and } \beta > 0$$

In the two parameter Mittage-Leffler function if we put $\alpha = \beta = 1$ then we get an exponential function.

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

Now we move towards the definitions of fractional derivative.

4. Popular Definitions of Fractional Derivative and Integration

Let α be any complex number and n be a natural number such that $n-1 < \mathbb{R}(\alpha) < n$, where $\mathbb{R}(\alpha)$ denotes the real part of α

Grunwald-Letnikov left sided derivative:

$${}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor n \rfloor} (-1)^k \frac{\Gamma(\alpha+1)\, f(t-kh)}{\Gamma(k+1)\,\Gamma(\alpha-k+1)} \quad , \quad where\ nh = t-a$$

Grunwald-Letnikov right sided derivative:

$${}_bD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor n \rfloor} (-1)^k \frac{\Gamma(\alpha+1)\, f(t+kh)}{\Gamma(k+1)\,\Gamma(\alpha-k+1)} \quad , \quad where\ nh = b-x$$

Riemann-Liouville left sided derivative:

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau$$

Riemann-Liouville right sided derivative:

$${}_bD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau$$

Riemann-Liouville fractional forward integral:

$${}_aD_t^{-\alpha} f(t) = {}_aI_t^\alpha = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

Riemann-Liouville fractional backward integral:

$${}_tD_b^{-\alpha} f(t) = {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau) d\tau$$

Weyl's fractional forward integral:

$${}_{-\infty}W_t^{-\alpha} f(t) = {}_{-\infty}I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-u)^{\alpha-1} f(u) du$$

Weyl's fractional backward integral:

$${}_tW_\infty^{-\alpha} f(t) = {}_tI_\infty^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (u-t)^{\alpha-1} f(u) du$$

Caputo left sided derivative:

$${}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau$$

Caputo right sided derivative:

$${}_b^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} f^{(n)}(\tau) d\tau$$

Oldham and Spanier derivative:

$$\frac{d^q}{dx^q} f(\beta t) = \beta^q \frac{d^q}{d(\beta t)^q} f(\beta t)$$

K.S. Miller and B. Rossderivative:

$$D^\alpha f(t) = D^{\alpha_1} D^{\alpha_2} \dots \dots \dots . D^{\alpha_n} f(t)$$

where$\alpha = \alpha_1 + \alpha_2 + \cdots \dots . . + \alpha_n$, $with\ \alpha_i < 1$

Kolwankar and Gangal derivative:

For $0 < q < 1$, the local fractional derivative at point $x = y$ for the function $f: [0,1] \to \mathbb{R}$ is given by,

$$D^q f(y) = \lim_{x\to y} \frac{d^q\big(f(x) - f(y)\big)}{d(x-y)^q}$$

5. Conclusion

There are some gaps in classical calculus and these gaps can be filled by fractional calculus. Recently serious efforts have been made in the detailed study of fractional calculus and its applications. Fractional differential equations arises in mathematical modeling of a physical phenomenon in the field of physics, chemistry, biology, controlled theory, signal and image processing, bio-chemistry, polymer theory etc. Most of the applied problem needs fractional derivative with proper utilization of initial conditions with known physical interpretation especially in the theory of viscoelasticity and solid mechanics. In such cases Caputo approach is more applicable because in Caputo fractional derivative the initial conditions are same as that of integer ordered differential equation and these initial conditions have known physical interpretation of the problem. Since fractional calculus has a wide interest for applications in different areas of physics and engineering, it has a great potential of integrating and presenting and has major applications in future.

6. Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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