A Matrix Trace Inequality for Products of Quaternion Hermitian Matrices

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Abstract: In this paper, the following matrix trace inequality for products of quaternion hermitian matrices A and B, \( \text{tr}(A \times B)_{2k} \leq \text{tr}(A^{2k} \times B^{2k}) \) is established, where k is a positive integer.

Keywords: Triple representation of quaternion matrix, Hermitian matrix, product, matrix trace inequality.

1. Introduction

The triple representation of complex matrices form a quaternion matrix, some new concept to quaternion division algebra where presented. [3]

The following matrix inequality for products of quaternion hermitian matrices A and B, \( \text{tr}(A \times B)_{2k} \leq \text{tr}(A^{2k} \times B^{2k}) \) is established, where k is a positive integer. [1]

Recently, there has been substantial interest in matrix trace inequalities for triple representation of complex and also hermitian matrices of the same order.[1,2]

2. Lemmas and Theorem

Lemma 2.1: Suppose that P is a quaternion square matrix; then

\[
\text{tr}[P^{2n-r} \times (P^H)^{2n-r}] \leq \text{tr}\left\{ \sum_{r=0}^{n-1} (P^{2n-r} \times P^H)^{2r} \right\} \text{tr}[P^2] 
\]

where n and r are integers.

Proof:

\[
\text{tr}[P^{2n-r} \times (P^H)^{2n-r}] = \text{tr}\left\{ \sum_{r=0}^{n-1} (P^{2n-r} \times P^H)^{2r} \right\} \text{tr}[P^2] 
\]

From equation (3) it is easy to know that \( S_{n+1} \) is quaternion hermitian and

\[
0 \leq \text{tr}S_{n+1}^2 = 2\text{tr}(P \times P^H)_{2n+1} - 2\text{tr}[(P \times P^H)^{2n} \times (P^H \times P)^{2n}]
\]

That is,

\[
\text{tr}[P \times P^H]_{2n+1} \leq \text{tr}[(P \times P^H)^{2n} \times (P^H \times P)^{2n}]
\]

Noticing that

\[
\text{tr}[P \times P^H]_{2n+1} \leq \text{tr}[(P \times P^H)^{2n} \times (P^H \times P)^{2n}]
\]

And combining the above equalities and inequality (4), we have

\[
\text{tr}[(P \times P^H)^{2n+1}] \geq \text{tr}[(P \times P^H)^{2n+1} \times (P^H \times P)^{2n+1}]
\]

Let

\[
R_{n-1} = (P \times P^H)^{2n-1} \times (P^H \times P)^{2n-1}
\]

Then it is easy to verify that the matrix \( R_{n-1} \) in (6) is quaternion hermitian, and by inequality (5) and (6), we have

\[
\text{tr}(P \times P^H)^{2n+1} \geq \text{tr}(R_{n-1}^2)
\]
Making use of Lemma 2.1 and inequality (7), we have
\[ \text{tr}(P \ast P^H)^{2k+1} \geq \text{tr}(R_{n-2}^2) = \text{tr}(R_{n-2}^2 \ast R_{n-2}^H)^2 \] (8)

Making use of the induction assumption and inequality (8), we have
\[ \text{tr}(P \ast P^H)^{2k+1} \geq \text{tr}[(R_{n-2}^2 \ast (R_{n-2}^H)^2] \]

Furthermore,
\[ \text{tr}(P \ast P^H)^{2k+1} \geq \text{tr}(R_{n-2}^2) \]

Since \( R_{n-2} \) is quaternion hermitian. Repeating the above solution, we have
\[ \text{tr}(P \ast P^H)^{2k+1} \geq \text{tr}[(P \ast P^H)^{2k}] \]

Furthermore,
\[ \text{tr}(P \ast P^H)^{2k+1} \geq \text{tr}[P^2 \ast (P^H)^2] \] (9)

Making use of the induction assumption, we have
\[ \text{tr}[P^2 \ast (P^H)^2] \geq \text{tr}[(P^2)^{2k} + [(P^H)^2]^{2k}] \]

Combining the inequalities (9) and (10), we have proved that Lemma 2.2 holds when \( k = n+1 \).

The Proof is complete.

**Theorem 2.3**: Suppose that \( A \) and \( B \) are quaternion Hermitian matrices of the same order; then
\[ \text{tr}(A \ast B)^{2k} \leq \text{tr}(A^2 \ast B^2) \]

where \( k \) is a positive integer.

**Proof**: When \( k = 1 \), it is easy to verify that the matrix \( A \ast B - A \ast B \) is skew-hermitian, and \( \text{tr}(A \ast B - A \ast B)^2 \leq 0 \) (12)

On the other hand, through direct calculation, we have
\[ (A \ast B - A \ast B)^2 = (A \ast B)^2 + (B \ast A)^2 - (A \ast B^2 \ast A) - (B \ast A^2 \ast B) \]

and
\[ \text{tr}(A \ast B - A \ast B)^2 = 2\text{tr}(A^2 \ast B^2) \]

Combining inequality (11) and (12), we have
\[ \text{tr}(A \ast B)^2 \leq \text{tr}(A^2 \ast B^2) \]

Hence Theorem 2.3 holds when \( k = 1 \).

Suppose that Theorem 2.3 holds when \( K \leq 1 \); in the following, we will prove that Theorem 2.3 is valid when \( k = n+1 \).

It is easy to verify that the matrix
\[ (A \ast B)^{2k} - [(A \ast B)^2]^{2k} \]

is quaternion skew hermitian, on the other hand, through direct calculation, we have,
\[ 0 \geq \text{tr}[(A \ast B)^{2k} - [(A \ast B)^2]^{2k}] \]
\[ = 2\text{tr}(A \ast B)^{2k+1} - 2\text{tr}[(A \ast B)^2 \ast [(A \ast B)^H]^{2k}] \]

Thus
\[ \text{tr}(A \ast B)^{2k+1} \leq \text{tr}[(A \ast B)^{2k} \ast [(A \ast B)^H]^{2k}] \] (14)

Making use of Lemma 2.2, we have
\[ \text{tr}[(A \ast B)^2 \ast [(A \ast B)^H]^{2k}] \leq \text{tr}[(A \ast B) \ast (A \ast B)^H]^{2k} = \text{tr}(A^2 \ast B^2) \] (15)

Making use of induction assumption, we have
\[ \text{tr}(A^2 \ast B^2)^{2k} \leq \text{tr}[(A^2)^{2k} \ast (B^2)^{2k}] = \text{tr}(A^2 \ast B^2)^{2k+1} \] (16)

Combining the inequalities (14) – (16), We have
\[ \text{tr}(A \ast B)^{2k+1} \leq \text{tr}(A^2 \ast B^2)^{2k+1} \] (17)

Thus we have proved that theorem 2.3 holds when \( k = n+1 \). The proof is complete.

**References**


Appl.188, 999-1001 (1994)


