We start this section by definition:
give some basic properties of this class of submodules.
In this section, we introduce the concept of the R
2.
See \[\text{notations N}\]
annihilator
The submodule
finitely
Jacobson radical and the singular submodule when M is
M is R-annihilator-small and \( K_M = A_M \), \( J(M) \subseteq A_M \) and \( Z(M) \subseteq A_M \) are given.

Keywords: hollow modules, annihilators, R-annihilator- hollow modules

1. Introduction

Throughout this paper all rings are associative ring with
identity and modules are unitary left modules. In [1] ,
Nicholson and Zhou defined annihilator-small right(left)
ideals as follows : a left ideal \( A \) of a ring R is called
annihilator-small if \( A+T=R \), where \( T \) is a left ideal , implies that \( r(T)=0 \) , where \( r(T) \) indicates the annihilator.
Kalati and.Keskin consider this problem for modules in [2] as follows:- let M be an R- module and \( S=\text{End}(M) \). A
submodule \( K \) of M is called annihilator-small if \( K+T =M \), \( T \) a submodule of M ,implies that \( r_S(T)=0 \) , where \( r_S \)
indicates the right annihilator of \( T \) over \( S= \text{End}(M) \), where \( r_S(T) = \{ f \in S ; f(T) = 0 , \forall t \in T \} \).

These observations lead us to introduce the following concept. A non-zero module M ,is called
R-annihilator –hollow module if every proper submodule of M is R-annihilator –small submodule of M.

In fact, the set \( K_M \) of all elements \( k \) such that \( Rk \) is
semisubmodule and annihilator-small. And contains both the
Jacobson radical and the singular submodule when M is
finitely generated and faithful.

The submodule \( A_M \) generated by \( K_M \) is a submodule of M
analogue of the Jacobson radical that contains every R-
annihilator-small submodules. In this work we give some
basic properties of
R-annihilator-hollow modules and various.

Characterizations
We abbreviate the Jacobson radical as \( \text{Rad}(M) \) and the
singular submodule as \( Z(M) \) for any R- module M. The
notations \( N \leq M \) mean that a submodule N of M is essential
in the module M.
See [1] / [2].

2. R- annihilator Small submodules

In this section, we introduce the concept of the R-annihilator
-small submodule and we illustrate it by examples. We also
give some basic properties of this class of submodules.
We start this section by definition:

Definition (2.1):
We say that a submodule N of an R-module M is a R-
annihilator-small submodule (R-a-small) if whenever
\( N+T=M \), \( T \) is a submodule of M, implies that \( \text{Ann}_M(T)=0 \), where \( \text{Ann}_M(T)= \{ r \in R ; r \cdot T =0 \} \). Clearly \( \text{Ann}_M(T) \) is a
left ideal of R. We write N\( \triangleleft \)M, see [3].

Let I be an ideal of a ring R. We say that I is R-a-small ideal of R if I is R-a-small submodule of R as an R-module.

Examples (2.2):
1) For an R- module M, M is not R-a-small submodule of M, where \( M=M+0 \) and \( \text{ann} 0 = \{ r \in R ; \ r \cdot 0 =0 \}=R \neq 0 \).
2) Let R be a commutative ring and I be an ideal of R. Then one can easily show that I is a small ideal of R if and only if I is R-a-small ideal of R as R-module, where \( r(1)=\text{ann}(I) \) when R is a commutative ring.

1) Let N be a submodule of an R-module M and let \( S= \text{End}(M) \). If N is a submodule of M then need not be N is R-a-small submodule of M as the following example shows:

Consider the module \( Z_6 \) as Z-module. \( \{ 0 \} \) is a small submodule of \( Z_6 \) and hence \( \{ 0 \} \) is a-small submodule of \( Z_6 \), by remark (1.3.4). But \( Z_6=\{ 0 \}+Z_6 \) and \( \text{ann} Z_6=\{ n \in \mathbb{Z} ; n \cdot Z_6=0 \} = 6Z \neq 0 \). Thus \( \{ 0 \} \) is not Z-a-small submodule of \( Z_6 \), See[7].

2) It is known that a non-zero small submodule can not be a direct summand. But this is not true for R-a-small submodules. For example, consider the module \( M=Z_2 \oplus Z_2 \) as \( Z_2 \)-module and

let A=\( Z_2 \oplus 0 \).
Clearly \( M=A \oplus Z_2= A \oplus (1_1,1_1) \) and \( \text{ann} 0 \oplus Z_2 = \text{ann}(1_1,1_1)=0 \). Thus A is
Z-a-small submodule of M.

The following three Corollary give more properties of R-a-small submodules.

Corollary (2.3):
Let K and N be a submodules of an R- module M such that
\( K \leq N \). If \( \frac{N}{K} \) is R-a-small submodule of \( \frac{M}{K} \), then N is R-a-small submodule of M.
Proof:
Let N, K be submodules of an R-module M such that K ≤ N and
\( \frac{N}{K} \) is R-a-small submodule of \( \frac{M}{K} \). Let \( \pi: M \to \frac{M}{K} \) be the natural epimorphism. Therefore \( \pi^{-1}(\frac{N}{K}) \) is R-a-small submodule of M, by prop (2.1.5). But \( \pi^{-1}(\frac{N}{K}) = N \). Thus N is R-a-small of M. See[7].

Corollary (2.4):
Let M be an R-module and let \( K \leq N \leq L \leq M \) such that \( \frac{L}{N} \) is R-a-small submodule of \( \frac{M}{N} \) then \( \frac{L}{K} \) is R-a-small submodule of \( \frac{M}{K} \).

Proof:
Let \( f: \frac{M}{K} \to \frac{M}{N} \) be the map defined by \( f(x+K) = x+K \), \( \forall x \in M \). One can easily to show f is an epimorphism. Since \( \frac{L}{N} \) is R-a-small submodule of \( \frac{M}{N} \), therefore \( \frac{L}{K} = f^{-1}(\frac{L}{N}) \) is R-a-small submodule of \( \frac{M}{K} \), by prop(2.1.5) Thus \( \frac{L}{K} \) is R-a-small submodule of \( \frac{M}{K} \). See[7].

Corollary (2.5):
Let M be an R-module and let \( K \leq N \leq M \), \( K' \leq N' \leq M \), if \( \frac{N+K}{K+K'} \) is R-a-small submodule of \( \frac{M}{K+K'} \). Then:-
1- \( \frac{N}{K} \) is R-a-small submodule of \( \frac{M}{K} \).
2- \( \frac{N+K}{K+K'} \) is R-a-small submodule of \( \frac{M}{K+K'} \).
3- \( \frac{N}{K} \oplus \frac{N'}{K} \) is R-a-small submodule of \( \frac{M}{K} \oplus \frac{M}{K} \).

Proof:
1- Let \( f_1: \frac{M}{K} \to \frac{M}{K+K'} \) be a map defined by \( f_1(x+K') = x+K' \), \( \forall x \in M \) and let \( f_2: \frac{M}{K+K'} \to \frac{M}{K+K'} \) be a map defined by \( f_2(m+K') = m+K' \), \( m \in M \). One can easily to show that each of \( f_1 \), \( f_2 \) is an epimorphism. Since \( \frac{N+K}{K+K'} \leq \frac{N+N'}{K+K'} \) and \( \frac{N+K}{K+K'} \) is R-a-small submodule of \( \frac{M}{K+K'} \), then \( \frac{N}{K} \) is R-a-small submodule of \( \frac{M}{K} \), by prop(2.1.4). Therefore \( \frac{N}{K} = f_1^{-1}(\frac{N+K}{K+K'}) \) is R-a-small submodule of \( \frac{M}{K} \), by prop (2.1.5)

2- Also \( \frac{N+K}{K+K'} \leq \frac{N+K}{K+K'} \) and \( \frac{N+K}{K+K'} \) is R-a-small submodule of \( \frac{M}{K+K'} \). Therefore \( \frac{K+N}{K} = f_2^{-1}(\frac{K+N}{K+K'}) \) is R-a-small submodule of \( \frac{M}{K} \), by prop (2.1.5)

3- From (1) and (2), we have \( \frac{N}{K} \oplus \frac{N}{K} \) is R-a-small submodule of \( \frac{M}{K} \oplus \frac{M}{K} \), by prop (2.1.8).

Let R be an integral domain. Recall that an R-module M is called a torsion free R-module if \( \text{ann}(x) = 0 \), for every non-zero element x in M, see[4].

Proposition (2.6):
Let M be a faithful R-module. Then every small submodule of M is R-a-small.

Proof:
Let M be faithful R-module and let N be a small submodule of M. To show N is R-a-small submodule of M. Let M = N+U. Since N is small in M, then M=U and hence annM = annU. So annU=0. Thus N is R-a-small submodule of M. The following corollary follows immediately of proposition (2.6).

Corollary (2.7):
Let R be an integral domain and let M be a projective R-module. Then every proper submodule of M is R-a-small submodule of M.

In particular, every proper submodule of a free module over an integral domain is R-a-small.

Remark (2.8):
Let M be an R-module if there exists a submodule N of M such that N is R-a-small submodule of M. Then M is faithful.

Proof:
Since M=N+M and N is R-a-small submodule of M, then \( \text{ann} M = 0 \). Thus M is faithful.

Proposition (2.9):
Let I be an ideal of a commutative ring R and let M be an R-module if I M is R-a-small in M, then I is a-small ideal of R.

Proof:
Let R=I+J, where I is an ideal of R. Then M=RM = (I+J)M = I+J M. Since I M is R-a-small in M, then \( \text{ann} J = 0 \). But \( \text{ann} J \leq \text{ann} J M \). Therefore \( \text{ann} J = 0 \). Thus I is R-a-small in R.

Recall that an R-module M is called a multiplication module if for every submodule N of M, there exists an ideal I of R such that \( N=IM \). Equivalently, an R-module M is a multiplication module if and only if \( N=IM \), for every submodule N of M, where \( (N:M) = \{ r \in R ; rM \leq N \} \). See [5].

We end the section by the following corollary and proposition we give various characterizations of R-annihilator-small submodules.

Corollary (2.10):
Let M be a multiplication module over a commutative ring R and let N be a submodule of M. If N is R-a-small submodule of M, then \( (N:M) \) is a-small ideal of R.

Proposition (2.11):
Let M be a module and K an R-a-small submodule of M. If \( \text{Rad}(M) \) is a small submodule of M and \( Z(M) \) is finitely
generated, then $K + \text{Rad}(M) + Z(M)$ is R-a-small submodule of $M$.

**Proof:**
Let $Z(M) = R z_1 + R z_2 + \ldots + R z_n$, where $z_i \in Z(M) \; \forall i=1,2,\ldots,n$.
To show $K + \text{Rad}(M) + Z(M)$ is R-a-small submodule of $M$, let $K + \text{Rad}(M) + Z(M) + X = M$, where $X$ is a submodule of $M$. Since $\text{Rad}(M)$ is a small submodule of $M$, then $K + Z(M) + X = M$. But $K$ is a small submodule of $M$, therefore $\text{ann}(Z(M) + X) = \text{ann}(R z_1 + R z_2 + \ldots + R z_n + X) = 0$. So $(\bigcap_{i=1}^{n} \text{ann}(R z_i)) \cap \text{ann}X = 0$. Since $z_i \in Z(M)$, then $\text{ann}(z_i) \subseteq R$. But $(\bigcap_{i=1}^{n} \text{ann}(R z_i)) \cap R = \{0\}$ by [12]. So $\text{ann}X = 0$. Thus $K + \text{Rad}(M) + Z(M)$ is R-a-small submodule of $M$.

3. **R- annihilator hollow modules**

In this section, we introduce the concept of the R-annihilator-hollow module and we study the basic properties of this type of module.

We start this section by definition:

**Definition (3.1):**
Non-zero module $M$ is called an R-annihilator-hollow module (R-a-hollow) if every proper submodule of $M$ is R-annihilator-small submodule of $M$.

**Examples (3.2):**
1- An R-a-small submodule of an R-module $M$ need not be small submodule.

For example, consider the module $Z$ as $Z$-module. For every $n \geq 1$, claim that $nZ$ is a Z-small submodule of $Z$. To show that, let $Z = nZ + nZ$, where $nZ$ is a submodule of $Z$. Since $Z$ has no-zero divisors, then $nZ = \{r \in Z : nr = 0\} = 0$. Thus $nZ$ is a Z-small submodule of $Z$. But it is known that $\{0\}$ is the only small submodule of $Z$, therefore $Z$ as $Z$-module is an R-annihilator-hollow module.

2- A small submodule of an R-module $M$ need not be R-a-small submodule. For example, consider $Z_4$ as $Z$-module. One can easily show that $\{0\}$ and $\{0,2\}$ are small submodules of $Z_4$. But $Z_4 = \{0\} + Z_4$ and $Z_4 = \{0,2\} + Z_4$ and $\text{ann}(Z_4) = \{nZ \in Z : n, Z_4 = 0\} = 4Z \neq 0$. Thus each of $\{0\}$ and $\{0,2\}$ is not a Z-small submodule of $Z_4$, therefore $Z_4$ is not an R-annihilator-hollow module.

3- Let $Z_p^\infty = \{x \in \mathbb{Q}_p : x \geq \frac{1}{p^n} + Z \}$ for some $r \in Z, n \in \mathbb{N}, p$ prime $\leq \frac{1}{p^n}$. $Z \leq \frac{1}{p^n} + Z$ is R-a-small submodule of $Z_p^\infty$ if $\{0\}$ is an R-annihilator-small submodule of $Z_p^\infty$ then $\text{ann}(Z_p^\infty) = \{z \in Z : n, Z_p^\infty = 0\} = 0$, therefore $\{0\}$ is an R-annihilator-small submodule of $Z_p^\infty$, $\frac{1}{p^n} + Z > Z_p^\infty$ and $Z_p^\infty = Z_p^\infty$, then $\text{ann}(Z_p^\infty) = 0$.

4- Let $\mathbb{N} \leq \mathbb{Q}$ and $N$ is R-annihilator-small submodule of $\mathbb{Q}$. Let $L \leq \mathbb{Q}$ such that $Q = N + L$ then $\text{ann}(N) = 0$. $Q$ is a torsion free then $\text{ann}(N) = 0$ for all $A \leq \mathbb{Q}$, therefore $N$ is R-annihilator-small submodule of $\mathbb{Q}$ then $Q$ as $Z$-module is R-annihilator-hollow module but $Q$ as $Z$-module is not hollow module.

The following two propositions give more properties of R-a-hollow module.

**Proposition (3.3):** Let $f: M \rightarrow M'$ be a homomorphism and let $M'$ be an R-a-hollow module such that for all $N \leq M$ such that $\text{Ker}f$ is small of $M$ then $M$ is an R-a-hollow module.

**Proof:**
Let $N \leq M$ with $M = N + K$, where $K$ is a submodule of $M$. To show $\text{ann}(K) = 0$.

**Proposition (3.4):** If $M = \frac{K}{K'}$ is an R-a-hollow module then $M$ is an R-a-hollow module for all $K$ submodule of $M$.

**Proof:**
Suppose $M = \frac{K}{K'}$ is an R-a-hollow module and let $N \leq M$ such that $M = N + L$, then $\text{ann}(L) = 0$. If $\text{ann}(K') = 0$, then $\text{ann}(L)$ is an R-a-hollow submodule of $K$, and $f(M) = N + f(L)$ is a torsion free module which is a contradiction. Therefore $N \leq M$. Since $f(N)$ is an R-a-hollow submodule of $M$, then $f(N) \leq M$. Thus $M$ is an R-a-hollow module. But $\text{ann}(K') = 0$ and $\text{ann}(K) \leq \text{ann}(K)$, therefore $\text{ann}(K) = 0$, and $N$ is an R-a-hollow submodule of $M$. Thus $M$ is an R-a-hollow module.

References