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Numerical Study of One - Dimensional Ground Water Recharge through Porous Media with Linear Permeability

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Abstract: ***The present paper deals with the approximate solution of one dimensional ground water recharge problem through porous media with linearpermeability. The phenomenon is formulated using Gauss Elimination Method. . This method is based on Lagrange multipliers for identification of optimal values of parameters in a functional. Using this method creates a sequence which tends to the exact solution of the problem. The Gauss Elimination Method (GEM) has been shown to solve effectively, easily and accurately a large class of linear problem with approximations converging rapidly to exact solutions. The solution of nonlinear partial differential equation by using finite element method, is in the term of ascending series and it is obtained by using Matlab and Mathematica.***

Keywords: Instabilities, Porous media, Partial Differential equation, Finite element method

1. Introduction

The unsteady and unsaturated flow of water through soils is due to content changes as a function of time and the entire pore spaces are not completely filled with flowing liquid respectively. Such type of flows helps some workers like hydrologist, agriculturalists, many fields of science and engineering. The water infiltrations system and the underground disposal of seepage and waste water are encountered by these flows, which are described by nonlinear partial differential equation.

The mathematical model conforms to the hydrological situation of one dimensional vertical ground water recharge by spreading [1]. Such flow is of great importance in water resource science, soil engineering and agricultural sciences. In this chapter, we have obtained a numerical solution of the problem by the finite element technique.

2. Statement of the problem

In the investigated mathematical model, we consider that the groundwater recharge takes place over a large basin of such geological location that the sides are limited by rigid boundaries and the bottom by a thick layer of water table. In this case, the flow is assumed vertically downwards through unsaturated porous media.

It is assumed that the diffusivity coefficient is equivalent to its average value over the whole range of moisture content, and the permeability of the media is either linear or parabolic function of the moisture content. The theoretical formulation of the problem yields a nonlinear partial differential equation for the moisture content.

3. Mathematical Formulation of the problem

We derive a mathematical model of one dimensional ground water recharge through porous media. The equation of continuity for an unsaturated porous medium [4], is given by

$$\frac{\partial}{\partial t}(\rho_s\theta) = -\nabla.M \quad \text{………….. (1)}$$

Where θ is moisture content on a dry weight basis, ρ_s is the bulk density of the medium and M is the mass flux of moisture.

From Darcy's law [1, 2, 3] the equation for the motion of water in a porous medium is,

$$V = -k\nabla\phi \quad \text{………….. (2)}$$

Where, V is the volume of the flux of moisture, $\nabla\phi$ - the gradient of the whole moisture potential and k the coefficient of aqueous conductivity. Combining equations (1) and (2), we get

$$\frac{\partial}{\partial t}(\rho_s\theta) = \nabla.(\rho k\nabla\phi) \quad \text{………….. (3)}$$

Where, ρ is the fluid density. Since, in the present case, we consider that the flow takes place only in the vertical direction, equation (3) reduces to,

$$\rho_s\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(\rho k\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial z}(\rho kg) \quad \text{………….. (4)}$$

Where ψ the capillary pressure potential, g is the gravitational constant and $\varphi = \psi - gz$ [5-9] The positive direction of the z-axis is the same as that of the gravity.

Considering θ and ψ to be connected by a single valued function, we may write (4) as,

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial\theta}{\partial z}\right) - \frac{\rho}{\rho_s}g\frac{\partial k}{\partial z} \quad \text{………….. (5)}$$

Where $D = \frac{\rho}{\rho_s} k \frac{\partial \psi}{\partial \theta}$ which is called diffusivity coefficient.

Now we assume permeability is a linear function of the moisture content, that is $k = k_o\theta$, $k_o = 0.232$, and replacing D by its average value D_a .we get

$$\frac{\partial \theta}{\partial t} = D_a \frac{\partial^2 \theta}{\partial z^2} - \frac{\rho}{\rho_s} k_o \frac{\partial \theta}{\partial z} \quad \ldots\ldots\ldots\ldots\ (6)$$

Now consider the water table to be situated at a depth L, and take $\frac{z}{L} = \xi$, $\frac{tD_a}{L^2} = T$, $\beta_o = \frac{\rho}{\rho_s}\frac{k_o}{D_a}$ we may write the boundary value problem as,

$$\frac{\partial \theta}{\partial T} = \frac{\partial^2 \theta}{\partial \xi^2} - \beta_o \frac{\partial \theta}{\partial \xi} \quad \ldots\ldots\ldots\ldots\ (7)$$

Where ξ = Penetration depth (dimensionless)

T = time (dimensionless)

β_o = Flow parameter (cm^2)

ρ = Mass density of water (gm)

ρ_s = Bulk density of the medium on dry weight basis ($gmcm^{-3}$)

k_o= Slope of the permeability vs moisture content plot ($cmsec^{-1}$)

D_a= Average value of the diffusivity coefficient over the whole range of moisture content (cm^2sec^{-1})

Now for definiteness, we choose a set of appropriate boundary conditions are

$$\theta(0,T) = \theta_o \quad , \quad \frac{\partial \theta}{\partial \xi}(1,T) = 0 \quad \ldots\ldots\ (8)$$

$$\theta(\xi, 0) = 0 \quad \ldots\ldots\ldots\ldots\ (9)$$

Where, initially we consider the moisture content throughout the region to be zero, at the layer z = 0 it is θ_o , and at the water table (z =L) it is assumed to remain 100% throughout the process of investigation. Note that the effect of capillary action at the stationary groundwater level, being very small, is neglected. The following values of the various parameters have been considered in the analysis: $\beta = b = 2.035$, $\theta_o = 0.5$, h =1/15 and k = 0.002223 for 225 time levels.

4. Mathematical Solution

We obtain the numerical solution of the equation (7) by finite element method. In the present problem the region of interest is the x – axis from ξ=0 to ξ=1. Suppose the region is divided into a set of n equal subinterval called element as discussed in 1.3.2. The elements are numbered as 1, 2, 3, … …, N, typical element being the e^{th} element of length h_e from node e to node e+1.

Now, the variational formulation of given partial differential equation (7) requires the functional

$$J(\theta) = \frac{1}{2}\int_R \left[\left(\frac{\partial \theta}{\partial \xi} \right)^2 + 2\theta \frac{\partial \theta}{\partial T} - \beta\theta \frac{\partial \theta}{\partial \xi} \right] d\xi \text{ is minimum} \quad \ldots\ldots\ldots\ldots\ (10)$$

We assume that the functional $J(\theta)$ can be written as the sum of N elemental functional as, $J(\theta^{(e)})$ as,

$$J = J(\theta) = \sum_{e=1}^{N} J\left(\theta^{(e)}\right) = \sum_{e=1}^{N} J^{(e)}$$

$$\text{Where } J^{(e)} = J\left(\theta^{(e)}\right) = \frac{1}{2}\int_{R^{(e)}} \left[\left(\frac{\partial \theta^{(e)}}{\partial \xi} \right)^2 + 2\theta^{(e)} \frac{\partial \theta^{(e)}}{\partial T} - \beta\theta^{(e)} \frac{\partial \theta^{(e)}}{\partial \xi} \right] d\xi \quad \ldots\ldots\ldots\ldots\ (11)$$

and $R^{(e)} = \left[\theta_1^{(e)}, \theta_2^{(e)}\right]$ is the domain of the typical element (e). The function $\theta^{(e)}$ is defined over the element (e) and zero elsewhere. We take the approximate solution using Linear Lagrange interpolation method as,

$$\theta^{(e)}(\xi) = \sum_{j=1}^{2} N_j(\xi)\theta_j^{(e)} = N^{(e)}\varphi^{(e)} = \varphi^{(e)^T} N^{(e)^T} \quad \ldots\ldots\ldots\ldots\ (12)$$

Where $N^{(e)} = [N_1\ N_2]$ and $\theta^{(e)} = [\ \theta_1\theta_2]^T$.

For line segment elements, shape functions are

$$N_1 = \frac{\xi - \xi_1}{\xi_2 - \xi_1} \text{ and } N_2 = \frac{\xi_2 - \xi}{\xi_2 - \xi_1} \quad \ldots\ldots\ldots\ldots\ (13)$$

Since $\theta^{(e)}(\xi) = N^{(e)}\varphi^{(e)} = \varphi^{(e)^T} N^{(e)^T}$,

$$\frac{\partial \theta^{(e)}}{\partial \xi} = \frac{\partial N^{(e)}}{\partial \xi}\varphi^{(e)} = \varphi^{(e)^T} \frac{\partial N^{(e)^T}}{\partial \xi}$$

and hence $\left(\frac{\partial \theta^{(e)}}{\partial \xi} \right)^2 = \varphi^{(e)^T} \frac{\partial N^{(e)^T}}{\partial \xi} \frac{\partial N^{(e)}}{\partial \xi} \varphi^{(e)}$.

Therefore the equation (11) can be written as,

$$J^{(e)} = J\left(\theta^{(e)}\right) = \frac{1}{2}\int_{R^{(e)}} \varphi^{(e)^T} \left[\left(\frac{\partial N^{(e)^T}}{\partial \xi} \frac{\partial N^{(e)}}{\partial \xi} \right)\theta^{(e)} + 2\left(N^{(e)^T} N^{(e)}\right)\frac{\partial \theta^{(e)}}{\partial T} - \beta\left(\frac{\partial N^{(e)^T}}{\partial \xi} N^{(e)} \right)\varphi^{(e)} \right] d\xi \quad \ldots\ldots\ldots\ldots\ (14)$$

The conditions for extremizing the equation (14) with respect to $\varphi^{(e)}$ give the element equations for a typical element (e) as,

$$\frac{\partial J^{(e)}}{\partial \varphi^{(e)}}=\int_{R^{(e)}}\left[\left(\frac{\partial N^{(e)^T}}{\partial \xi}\frac{\partial N^{(e)}}{\partial \xi}\right)\theta^{(e)}+2\left(N^{(e)^T}N^{(e)}\right)\frac{\partial \theta^{(e)}}{\partial T}-\beta\left(\frac{\partial N^{(e)^T}}{\partial \xi}N^{(e)}\right)\varphi^{(e)}\right]d\xi=0$$

In matrix form, we may write the element equation as

$$A^{(e)}\frac{\partial \phi^{(e)}}{\partial T}+B^{(e)}\phi^{(e)}+C^{(e)}\phi^{(e)}=0 \text{ ………….. (15)}$$

where , $A^{(e)}$, $B^{(e)}$ and $C^{(e)}$ are called element matrix for the typical element (e) defined as,

$$A^{(e)}=\int_{\theta_1^{(e)}}^{\theta_2^{(e)}}\left(N^{(e)^T}N^{(e)}\right)d\xi ,$$

$$B^{(e)}=\int_{\theta_1^{(e)}}^{\theta_2^{(e)}}\left(\frac{\partial N^{(e)^T}}{\partial \xi}N^{(e)}\right)d\xi ,$$

$$C^{(e)}=\int_{\theta_1^{(e)}}^{\theta_2^{(e)}}\left(\frac{\partial N^{(e)^T}}{\partial \xi}\frac{\partial N^{(e)}}{\partial \xi}\right)d\xi$$

Now, for the evaluation of these integrals, we use Gauss Legendre Quadrature Method. So we transform co-ordinate system ξ to a local coordinate system in z such that for $\xi = \theta_1$, we get z = -1 and for $\xi = \theta_2$, we get z = 1. Therefore the shape function becomes, $N_1(z)=\frac{1}{2}(1-z)$ and $N_2(z)=\frac{1}{2}(1+z)$ and Jacobian matrix is $J=\frac{1}{2}h$. Now, by applying Gauss Legendre Quadrature method to the above integrals, we get the element matrix for the typical element (e) as,

$$A^{(e)}=\int_{-1}^{1}\left(N^{(e)^T}N^{(e)}\right)Jdz\approx\sum_{i=1}^{r}A^{(e)}(z_i)W_i$$

$$B^{(e)}=-\beta\int_{-1}^{1}\left(\frac{1}{J}\frac{\partial N^{(e)^T}}{\partial z}N^{(e)}\right)Jdz\approx\sum_{i=1}^{r}B^{(e)}(z_i)W_i$$

$$C^{(e)}=\int_{-1}^{1}\left(\frac{1}{J}\frac{\partial N^{(e)^T}}{\partial z}\frac{1}{J}\frac{\partial N^{(e)}}{\partial z}\right)Jdz\approx\sum_{i=1}^{r}C^{(e)}(z_i)W_i$$

Where Z_i and W_i are corresponding gauss points and gauss weights with respect to r which can be obtained from the table 1.1. We consider degree of polynomial p =2 for $A^{(e)}$ then r = 2 and p = 1 for $B^{(e)}$ and $C^{(e)}$ then r =1. Therefore the element matrix for the typical element (e) becomes,

$$A^{(e)}=\frac{h^{(e)}}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix},\quad B^{(e)}=\frac{-\beta}{2}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix},$$

$$C^{(e)}=\frac{1}{h^{(e)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \text{ ………….. (16)}$$

Now we express the element equations (15) in terms of the global nodal values θ_i for each element. Since the equation (15) is derived for an arbitrary typical element, it holds for any element from the finite element mesh. Since the elements are connected at nodes 2 and 3, 3 and 4, …….., N-1 and N and θ is continuous, θ_2 of e^{th} element should be the same as θ_1 of $(e + 1)^{th}$ element for e = 1, 2, ………, N and sum of these two vanishes if there is no external point source applied otherwise it is considered as value of the magnitude. The inter-element continuity of primary variable can be imposed by simply renaming the variables of all elements connected to common node. Now we consider a uniform mesh of N elements, then the assembled equation is obtained by equation (15) and (16) as

$$A\frac{\partial \varphi}{\partial T}-B\varphi+C\varphi=0 \text{ ………….. (17)}$$

where the assembled matrix becomes,

$$A=\frac{h}{6}\begin{bmatrix}2 & 1 & 0 & 0 & \dots & 0 & 0\\ 1 & (2+2) & 1 & 0 & \dots & 0 & 0\\ 0 & 1 & (2+2) & 0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & (2+2) & 1\\ 0 & 0 & 0 & 0 & \dots & 1 & 2\end{bmatrix}$$

$$B=\frac{-\beta}{2}\begin{bmatrix}-1 & 1 & 0 & 0 & \dots & 0 & 0\\ 1 & (1-1) & -1 & 0 & \dots & 0 & 0\\ 0 & 1 & (1-1) & 0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & (1-1) & 1\\ 0 & 0 & 0 & 0 & \dots & 1 & 1\end{bmatrix}$$

$$C=\frac{1}{h}\begin{bmatrix}1 & -1 & 0 & 0 & \dots & 0 & 0\\ -1 & (1+1) & -1 & 0 & \dots & 0 & 0\\ 0 & -1 & (1+1) & 0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & (1+1) & -1\\ 0 & 0 & 0 & 0 & \dots & -1 & 1\end{bmatrix}$$

$$\varphi = [\theta_1\theta_2 \quad \dots \quad \dots \quad \theta_{N+1}] \text{ ………….. (18)}$$

Now we introduced δ a family of approximations which approximates weighted average of a dependent variable of two consecutive time steps by linear interpolation of the values of the variable at the two time steps as,

$$\theta_j=\delta\theta_j^{n+1}+(1-\delta)\theta_j^{n} \text{ ………….. (19)}$$

The time derivatives θ_j are replaced by forward finite difference formula such as,

$$\theta_j = \frac{\theta_j^{n+1} - \theta_j^n}{k} \quad \ldots\ldots\ldots\ldots (20)$$

Hence the equation (2.2.1.8) can be written as,

$$\left\{A + \delta k (B+C)\right\} \varphi^{(n+1)} = \left\{A - (1-\delta)k(B+C)\right\} \varphi^{(n)} \quad \ldots\ldots\ldots\ldots (21)$$

where $\delta = 1/2$ and n = 0,1,2,…. ….

Using the assembled matrices (18), the global equation (21) takes the form,

$$M'\varphi^{(n+1)} = F'\varphi^{(n)} = F''\left(\varphi^{(n)}\right) \quad \ldots\ldots\ldots\ldots (22)$$

where N+1 is total number of global nodes, global stiffness matrix and global generalized force vector F" are defined as,

$$M' = \begin{bmatrix} m'_{11} & m'_{12} & 0 & \ldots & \ldots & 0 & 0 \\ m'_{21} & m'_{22} & m'_{23} & 0 & \ldots & 0 & 0 \\ 0 & m'_{32} & m'_{33} & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & m'_{NN} & m'_{N,N+1} \\ 0 & 0 & 0 & 0 & \ldots & m'_{N+1,N} & m'_{N+1,N+1} \end{bmatrix}$$

Where, $m'_{11} = \frac{h}{3} + \delta k\left(\frac{\beta}{2} + \frac{1}{2}\right)$,

$$m'_{12} = \frac{h}{6} + \delta k\left(\frac{\beta}{2} - \frac{1}{h}\right), m'_{21} = \frac{h}{6} - \delta k\left(\frac{\beta}{2} + \frac{1}{h}\right)$$

$$m'_{22} = \frac{2h}{3} + \delta k\left(\frac{2}{h}\right), m'_{23} = \frac{h}{6} + \delta k\left(\frac{\beta}{2} - \frac{1}{h}\right),$$

$$m'_{32} = \frac{h}{6} - \delta k\left(\frac{\beta}{2} + \frac{1}{h}\right)$$

$$m'_{33} = \frac{2h}{3} + \delta k\left(\frac{2}{h}\right), m'_{NN} = \frac{2h}{3} + \delta k\left(\frac{2}{h}\right),$$

$$m'_{N,N+1} = \frac{h}{6} + \delta k\left(\frac{\beta}{2} - \frac{1}{h}\right)$$

$$m'_{N+1,N} = \frac{h}{6} - \delta k\left(\frac{\beta}{2} + \frac{1}{h}\right),$$

$$m'_{N+1,N+1} = \frac{h}{3} + \delta k\left(\frac{\beta}{2} + \frac{1}{h}\right)$$

$$F'\left(\varphi^{(n)}\right) = \begin{bmatrix} f'_{11} & f'_{12} & 0 & \ldots & \ldots & 0 & 0 \\ f'_{21} & f'_{22} & f'_{23} & 0 & \ldots & 0 & 0 \\ 0 & f'_{32} & f'_{33} & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & f'_{NN} & f'_{N,N+1} \\ 0 & 0 & 0 & 0 & \ldots & f'_{N+1,N} & f'_{N+1,N+1} \end{bmatrix}$$

Where, $f'_{11} = \frac{h}{3} - (1-\delta)k\left(\frac{\beta}{2} + \frac{1}{h}\right)$,

$$f'_{21} = \frac{h}{6} + (1-\delta)k\left(\frac{\beta}{2} + \frac{1}{h}\right),$$

$$f'_{12} = \frac{h}{6} - (1-\delta)k\left(\frac{\beta}{2} - \frac{1}{h}\right)$$

$$f'_{22} = \frac{2h}{3} - (1-\delta)k\left(\frac{2}{h}\right), f'_{23} = \frac{h}{6} - (1-\delta)k\left(\frac{\beta}{2} - \frac{1}{h}\right),$$

$$f'_{32} = \frac{h}{6} + (1-\delta)k\left(\frac{\beta}{2} + \frac{1}{h}\right)$$

$$f'_{33} = \frac{2h}{3} - (1-\delta)k\left(\frac{2}{h}\right), f'_{NN} = \frac{2h}{3} - (1-\delta)k\left(\frac{2}{h}\right),$$

$$f'_{N.N+1} = \frac{h}{6} - (1-\delta)k\left(\frac{\beta}{2} - \frac{1}{h}\right)$$

$$f'_{N+1,N} = \frac{h}{6} + (1-\delta)k\left(\frac{\beta}{2} + \frac{1}{h}\right),$$

$$f'_{N+1,N+1} = \frac{h}{3} - (1-\delta)k\left(\frac{\beta}{2} + \frac{1}{h}\right)$$

$$F''\left(\varphi^{(n)}\right) = \left[f''_{ij}\right]$$

Where, $f'_{11} = \left\{\frac{h}{3} - (1-\delta)k\left(\frac{\beta}{2} + \frac{1}{h}\right)\right\}\theta_1^{(n)} + \left\{\frac{h}{6} - (1-\delta)k\left(\frac{\beta}{2} - \frac{1}{h}\right)\right\}\theta_2^{(n)}$

$$f'_{21} = \left\{\frac{h}{6} + (1-\delta)k\left(\frac{\beta}{2} + \frac{1}{h}\right)\right\}\theta_1^{(n)} + \left\{\frac{2h}{3} - (1-\delta)k\left(\frac{2}{h}\right)\right\}\theta_2^{(n)} + \left\{\frac{h}{6} - (1-\delta)k\left(\frac{\beta}{2} - \frac{1}{h}\right)\right\}\theta_3^{(n)}$$

$$f'_{31} = \left\{\frac{h}{6} + (1-\delta)k\left(\frac{\beta}{2} + \frac{1}{h}\right)\right\}\theta_2^{(n)} + \left\{\frac{2h}{3} - (1-\delta)k\left(\frac{2}{h}\right)\right\}\theta_3^{(n)}$$

… …

$$f'_{N,1} = \left\{\frac{2h}{3} - (1-\delta)k\left(\frac{2}{h}\right)\right\}\theta_N^{(n)} + \left\{\frac{h}{6} - (1-\delta)k\left(\frac{\beta}{2} - \frac{1}{h}\right)\right\}\theta_{N+1}^{(n)}$$

$$f'_{N+1,1} = \left\{\frac{h}{6} + (1-\delta)k\left(\frac{\beta}{2}+\frac{1}{h}\right)\right\}\theta_N^{(n)} + \left\{\frac{h}{3} - (1-\delta)k\left(\frac{\beta}{2}+\frac{1}{h}\right)\right\}\theta_{N+1}^{(n)}$$

Now we apply the boundary condition (8) to the global equation (22) and simplifying, we get

$M\varphi^{(n+1)} = F$ ………….. (23)

$$M = \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & 0 & 0 \\ m_{21} & m_{22} & m_{23} & 0 & \dots & 0 & 0 \\ 0 & m_{23} & m_{33} & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & m_{N,N} & m_{N,N+1} \\ 0 & 0 & 0 & 0 & \dots & m_{N+1,N} & m_{N+1,N+1} \end{bmatrix}$$

Where, $m_{21} = \frac{h}{6} - \delta k\left(\frac{\beta}{2}+\frac{1}{h}\right), m_{22} = \frac{2h}{3} + \delta k\left(\frac{2}{h}\right),$

$$m_{23} = \frac{h}{6} + \delta k\left(\frac{\beta}{2}-\frac{1}{h}\right)$$

$$m_{32} = \frac{h}{6} - \delta k\left(\frac{\beta}{2}+\frac{1}{h}\right), m_{33} = \frac{2h}{3} + \delta k\left(\frac{2}{h}\right),$$

$$m_{N,N} = \frac{2h}{3} + \delta k\left(\frac{2}{h}\right)$$

$$m_{N,N+1} = \frac{h}{6} + \delta k\left(\frac{\beta}{2}-\frac{1}{h}\right), m_{N+1,N} = \frac{h}{6} - \delta k\left(\frac{\beta}{2}+\frac{1}{h}\right)$$

$$, m_{N+1,N+1} = \frac{h}{3} + \delta k\left(\frac{\beta}{2}+\frac{1}{h}\right)$$

Where, $f_{11} = \theta_0$

$$f_{21} = \left\{\frac{h}{6} + (1-\delta)k\left(\frac{\beta}{2}+\frac{1}{h}\right)\right\}\theta_1^{(n)} + \left\{\frac{2h}{3} - (1-\delta)k\left(\frac{2}{h}\right)\right\}\theta_2^{(n)} + \left\{\frac{h}{6} - (1-\delta)k\left(\frac{\beta}{2}-\frac{1}{h}\right)\right\}\theta_3^{(n)}$$

$$f_{31} = \left\{\frac{h}{6} + (1-\delta)k\left(\frac{\beta}{2}+\frac{1}{h}\right)\right\}\theta_2^{(n)} + \left\{\frac{2h}{3} - (1-\delta)k\left(\frac{2}{h}\right)\right\}\theta_3^{(n)}$$

… …

$$f_{N,1} = \left\{\frac{2h}{3} - (1-\delta)k\left(\frac{2}{h}\right)\right\}\theta_N^{(n)} + \left\{\frac{h}{6} - (1-\delta)k\left(\frac{\beta}{2}-\frac{1}{h}\right)\right\}\theta_{N+1}^{(n)}$$

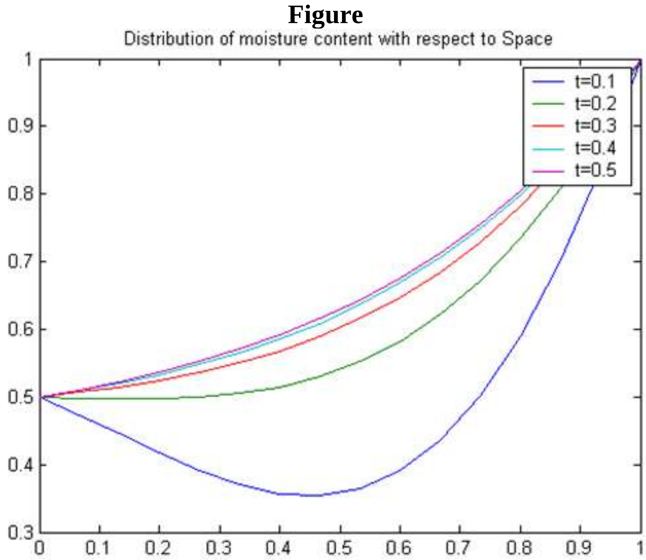
$$f_{N+1,N} = \left\{\frac{h}{6} + (1-\delta)k\left(\frac{\beta}{2}+\frac{1}{h}\right)\right\}\theta_N^{(n)} + \left\{\frac{h}{3} - (1-\delta)k\left(\frac{\beta}{2}+\frac{1}{h}\right)\right\}\theta_{N+1}^{(n)}$$

Hence we get N+1 algebraic equation in N+1 unknown which can be solved by Gauss elimination method. At the beginning of the iteration (i.e. n=0), we assume the solution $\varphi^{(0)}$ from initial condition (9) which requires to have $\theta_1^{(0)} = \theta_2^{(0)} = \dots \quad \dots = \theta_{N+1}^{(0)} = 0$. A Matlab Code is prepared for 15 elements model and resulting equation (23) for N = 15 is solved by Gauss elimination method. The numerical value are shown in the following table and plotted in figure given below. Curves indicating the behavior of moisture content corresponding to various time period have been shown in the figure.

Table: Moisture content at different time

	θ				
ξ	t = 0.1	t = 0.2	t = 0.3	t = 0.4	t = 0.5
0	0.5000	0.5000	0.5000	0.5000	0.5000
0.0666	0.4736	0.4985	0.5067	0.5095	0.5104
0.1332	0.4454	0.4974	0.5147	0.5205	0.5224
0.1998	0.4173	0.4976	0.5242	0.5332	0.5362
0.2664	0.3914	0.4998	0.5359	0.5480	0.5520
0.3330	0.3704	0.5052	0.5502	0.5652	0.5703
0.3996	0.3569	0.5149	0.5677	0.5854	0.5913
0.4662	0.3538	0.5301	0.5892	0.6090	0.6156
0.5328	0.3640	0.5521	0.6153	0.6365	0.6436
0.5994	0.3902	0.5822	0.6469	0.6686	0.6758
0.6660	0.4351	0.6218	0.6848	0.7059	0.7130
0.7326	0.5010	0.6720	0.7298	0.7492	0.7557
0.7992	0.5895	0.7340	0.7829	0.7993	0.8048
0.8658	0.7019	0.8087	0.8449	0.8571	0.8611
0.9324	0.8388	0.8972	0.9170	0.9236	0.9258
0.9990	1.0000	1.0000	1.0000	1.0000	1.0000

Figure



5. Conclusion

In above graph, X-axis represents the values of ξ and Y-axis represents moisture content (θ) of unsaturated porous media in large basin of length one. We consider that the sides of basin are limited by rigid boundaries and bottom at a thick layer of water table so that water flows only in positive direction. It is interpreted from the graph that as time increases, the moisture content also increases but the rate at which moisture content rises at each point in basin slows down with increase in time.

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