International Journal of Science and Research (IJSR)
ISSN: 2319-7064
Index Copernicus Value (2016): 79.57 | Impact Factor (2017): 7.296

Cubic Spline Extrapolation

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Abstract: ***The existence, uniqueness and convergence properties of extrapolated cubic spline***

1. Introduction

It has been observed that piecewise polynomials functions which satisfy a less stringent smoothness requirement than the maximum non trivial smoothness have also some interesting and useful properties (see Schumaker [9]). Spline interpolation is preferred over polynomial interpolation because the interpolation error can be made small even when using low degree polynomials for the spline. Spline interpolation avoids the problem of Runge's phenomenon, which occurs when the interpolating uses high degree polynomials [2].An important development in this direction and Schumaker [5] (See also Dubey [3], [4], [5] Rana and Dubey [8], Malcolm [7], Demkos [2], and Dikshit and Rana [6]). It is mentioned that continuous cubic spline may be used as a limiting case of the discrete cubic spline. In fact the defining condition for discrete cubic spline involves in some sense a certain process of extrapolation. We use this approach here for defining extrapolated deficient cubic splines. The class of all piecewise polynomial functions s_i of degree 3 or less which satisfy the condition,

$$(s_{i+1}-s_i)(x_i-jh)=0,\ \text{i = 1,2,.....n,} \tag{1.1}$$

for h > 0 and j = 0, 1 defines the class $S(3,P,h)$ of extrapolated deficient cubic splines. To be more specific we denote the elements of $S(3,P,h)$ by s^h. It may be mentioned that condition (1.1) is less stringent the corresponding condition used for defining discrete cubic splines. We shall study in the present paper existence, uniqueness and convergence property of extrapolated cubic spline with multiple knots which interpolate a given function at two points of a general choice of set of points interior to each mesh interval which includes some earlier results in this direction of particular choice.

We set for convenience

$u_i=x_{i-1}+(1/3)p_i$ and

$v_i=x_{i-1}+(2/3)\,p_i$ for i = 1,2,.....n

where (1/3) and 2/3 are real numbers and p_i is the length of mesh interval $[x_{i-1}, x_i]$ for the mesh P of [0, 1] given by $P:0=x_0<x_1<........x_n=1$ and

$p=\max_i p_i,\ p'=\min_i p_i$.

We propose to study the following:

Problem 1.1. Given functional values $\{f(u_i)\}$ and $\{f(v_i)\}$, to find the condition on θ and p which lead to a unique extrapolated deficient cubic splines satisfying the following interpolatory conditions :

$$s^h(u_i)=f(u_i) \tag{1.2}$$

$$s^h(v_i)=f(v_i) \tag{1.3}$$

for i = 1,2,......n

2. Existence and Uniqueness

In order to answer the problem 1.1, we set for convenience

$R_i(x)=(x-x_{i-1})(x-u_i)(x-v_i)$,

$Q_i(x)=(x-x_i)(x-u_i)(x-v_i)$,

$R_i(x,u)=(x-x_{i-1})(x-u_i)^2$

$Q_i(x,v)=(x-x_i)(x-v_i)^2$.

$R_i(x)$ with the factor $(x-x_{i-1})$ replaced by $(x-x_i)$ define $Q_i(x)$., we state the following equations which will be useful,

$R_i(x_i-h)=(1/3)(p_i-h)(p_i-3h)(2p_i-3h)$

$Q_i(x_i-h)=-(1/3)h(p_i-3h)(2p_i-3h)$

$R_i(x_i-h,u)=(4/9)(p_i-h)(2p_i-3h)^2$

and $Q_i(x_i-h,v)=-(1/9)h(p_i-3h)^2$.

We shall answer the problem 1.1 in the following.

Theorem 2.1. Suppose that f is 1 periodic and p' is such that for $h>0(i)\ p'\geq 3h$ or (ii) $<p_i>$ is a non increasing sequence with $p'\geq h$, i=1,2,.....n holds then there exist a unique 1 periodic spline $s^h\in S(3,P,h)$ which satisfies the interpolatory condition (1.2) and (1.3)

Proof of Theorem 2.1. It is clear that $s^h\in S(3,P,h)$ then we may write

$$S_i^h(x)=AQ_i(x)-B\,R_i(x)+C\,R_i(x,u)-D\,Q_i(x,v) \tag{2.1}$$

where A, B, C and D are constants to be determined.

$$f(u_i)=2Dp_i^3/27 \tag{2.2}$$

$$\text{and } f(v_i)=C\,P_i^3/27 \tag{2.3}$$

If we now set $s^h(x_i)=N_i(h)$, i = 0,1,2,.....n and use (2.2) and (2.3), then we have from (2.1).

$$N_i(h)=-(2/3)\,P_i^3\,B+(2/27)\,f(v_i)/ \tag{2.4}$$

$$\text{and } N_{i-1}(h)=3[-P_i^3A+f(u_i)] \tag{2.5}$$

Thus in view of (2.2) - (2.5) we see that for the interval $[x_{i-1}, x_i]$,

$(2/81)\, p_i^3\, s_i^h(x) = (1/9)[R_i(x)\, N_i h - Q_i(x)\, N_{i-1}(h)$

$$+ f(u_i)[(2/3)Q_i(x) - (1/3)Q_i(x,v)] + f(v_i)[(1/3)R_i(x,u) - (2/3)R_i(x)]. \quad (2.6)$$

Now it follows from (2.6) and (1.1) with j=1 that

$$h\,L(p_i,-h)\, p_{i+1}^3\, N_{i-1}(h) + [(p_i - h)\,L(p_i,-h)\, p_{i+1}^3 - (p_{i+1}+h)\,L(p_{i+1},h)\, p_i^3]$$

$$N_i(h) + h\,L(p_{i+1},h)\, p_i^3\, N_{i+1}(h) = F_i(1/3,h) \quad (2.7)$$

where $(1/3)\,F_i(1/3,h) = h\, p_i^3\,(p_{i+1}+h)\big[(1/3)p_{i+1}+h)\, f(v_{i+1}) - (2/3)\,p_{i+1}+h)\, f(u_{i+1})\big]$

$-h\, p_{i+1}^3\,(p_i - h)\big[(2/3)\,p_i - h)\, f(v_i) - (1/3)p_i - h)\, f(u_i)\big]$

and

$L(P_i, jh) = (1/3)\,p_i + jh)(\ 1/3)p_i + jh)$

for all i = j = 1, -1.

In order to prove theorem 1.2, it is sufficient to show that the system of equation (2.7) for i=1,2,....,n has a unique set of solutions. Clearly the coefficients of $N_{i+1}(h)$ is non-negative. Further in view of the condition (ii) of theorem 2.1 as $p_{i+1}/p_i \le 1$, we observe that the coefficients of $N_i(h)$ is non-positive and the absolute value of the coefficient of $N_{i-1}(h)$ is

$|\,h\,L(P_i,-h)\,P_{i+1}^3\,| < h\,L(P_i,h)\,p_{i+1}^3 = h(1/3)P_i + h)(2/3)p_i + h)$

Thus, the excess of the positive value of the coefficient of $N_i(h)$ over the sum of the positive value of the coefficients of $N_{i-1}(h)$ and $N_{i+1}(h)$ in (2.7) is less than $b_i(h) = h\,p_i\,p_{i+1}\left[p_i^2\,(p_{i+1}+h) + p_{i+1}^2(p_i - 3h)\right]$ which is clearly positive under the condition (i) or (ii) of Theorem 2.1. We thus conclude that the coefficient matrix of the system of equation (2.7) is diagonally dominant and hence invertible. This, completes the proof of Theorem 2.1.

3. Error Bounds

In this section of the present paper, we should estimate the bounds for the error function $e = s^h - f$, where s^h is the interpolaotry spline of Theorem 2.1. For convenience we assume in this section of this paper, that the mesh points are equidistant, so that

$p_i = p$, i = 0, 1,.....n.

We now introduce function $t^{(r)}$ which is the same as $(s^h)^{(r)}$. At the mesh points the function $t^{(r)}$, r=1,2,..... is defined by

$t^{(r)}(x_i) = (s_i^h)^{(r)}(x_i)$, i = 0,1,....n. (3.1)

It is of course clear that since $s^h \in C[0,1]$

Using the foregoing notation of $t^{(r)}(x)$, we shall prove the following.

Theorem 3.1. Suppose that f'' exist in [0,1] then for interpolatory spline s^h of theorem 2.1 we have

$$\|(t^{(r)} - f^{(r)})(x)\| \le \{(2/3)\}p)^{2-r}\, K(h,1/3)\, w(f''\, p)$$

for r=0,1,2,...... (3.2)

where $K(h,1/3)$ is a positive function of h and 1/3

Proof of Theorem 3.1. It may be observed that the system of equations (2.7) may be written as

$$A(h)\,N(h) = F(h), \quad (3.3)$$

where A(h) is the coefficient matrix having non zero element in each row.

N(h)= (Ni(h)) and F(h) denotes the single column matrix $(F_i(1/3,h))$. In view of the diagonal dominant property of A(h) (See Ahlberg, Nilson and Walsle [1]). It may be seen that

$$\|A^{-1}(h)\| \le a(h) \quad (3.4)$$

where $a(h) = \{2h\,p^4(p-h)\}^{-1}$

We rewrite the equation (3.3) to obtain

$$A(h)(N_i(h) - f_i) = F_i(1/3,h) - A(h)\,f_i \quad (3.5)$$

We first proceed to estimate the right hand side of (3.5). Applying the Taylor's theorem appropriately we observe that the i^{th} row of the right hand side of appearing in (3.5) is (3/2)

$$h\,p^5[(p+h)\{4/9)(1/3\,p+h)\,f''(\alpha_{i+1}) - (2/27)p+h)\,f''(\beta_{i+1})\}$$

$$-(p-h)\{1/9)(2/3p-h)\,f''(\alpha_i) - (4/27)p-h)\,f''(\beta_i)\}]$$

$$-h\,p^5[(1/3)p+h)(2/3\,p+h)\,f''(z_i) + (1/3)p-h)(2/3)(\,p-h)\,f''(\delta_{i+1})]$$

where α_i, β_i, z_i and $\delta_i \in [x_{i-1}, x_i]$ for all i. Now using (3.4) and adjusting suitably the terms of right hand side of (3.5), we have

$$\|(N_i(h) - f_i)\| \le p^2 K_1(l,1/3)\, w(f'', p) \quad (3.6)$$

where

$K_i(l,1/3) = (4/9 + (5/3)d + 4/3d^2)/h(1/3)(1-d)$

with $d = p/h$. Observing that

$R_i''(x_i) = 2Q_i''(x_i) = hp$

$Q_i''(x_i,v)=h(1/3)\,p$ and

$R_i''(x_i,u)=(14/3)p$, we have from (2.6)

$(2/27)p^2\left[(s_i^h)''(x_i)-f''(x_i)\right],$

$=(2/3)[2(N_i(h)-f_i)-(N_{i-1}(h)-f_{i-1})]+G_i(f)$ (3.7)

where $G_i(f)=98/3)\,f(u_i)-(10/3)\,f(u_i)+(4/3)\,f_i$ $-(2/3)\,f_{i-1}-(2/27)\,p^2\,f''(x_i)$.

By an appropriate application of Taylor's theorem, we have

$G_i(f)=p^2[(16/27)\,f''(\beta_i)-(5/27)\,f''(\alpha_i)$ $-(1/3)\,f''(\delta_i)-(2/9)\,f''(x_i)]$

where $\alpha_i,\beta_i,\delta_i\in[x_{i-1},x_i]$ for all i. Again adjusting suitably the terms of $G_i(f)$, we get

$$\|G_i(f)\|\le(34/27)p^2\,w(f'',p). \quad (3.8)$$

Then using (3.6) and (3.8), we have from (3.7)

$\|((s_i^h)''-f'')(x_i)\|\le K_2(h,1/3)\,w(f'',p)$

where $K_2(h,1/3)=[9K_1(l,1/3)+9]$

$(s_i^h)''$ is piecewise linear, so that for $[x_{i-1},x_i]$. (3.9)

$p(s_i^h)''(x)=$

$$(s_i^h)''(x_{i-1})(x_i-x)+(s_i^h)''(x_i)(x-x_{i-1}) \quad (3.10)$$

and hence,

$p((s_i^h)''-f'')(x)=(x_i-x)\left[(s_i^h)''(x_{i-1})-(s_{i-1}^h)''(x_{i-1})\right]+$ $+(x_i-x)\left[(s_{i-1}^h)''(x_{i-1})-f_{i-1}''\right]+(x-x_{i-1})\left[(s_i^h)''(x_i)-f_i''\right]$ $+(x_i-x)(f_{i-1}''-f''(x)+(x-x_{i-1})(f_i''-f''(x))$.

Thus, $\|((s_i^h)''-f'')(x)\|\le((s_i^h)''-(s_{i-1}^h)'')(x_{i-1})\|$ $+\|((s_i^h)''-f'')(x_i)\|+w(f'',p)$. (3.11)

Next, we see that

$2R_i''(x_{i-1})=Q_i''(x_{i-1})=-hp,$

$R_i''(x_{i-1},u)=-h(1/3)p$ and $Q_i''(x_{i-1}\,v)=-(14/3)p$

So in view of (2.6), we have

$(2/27)(p^2(S_i^h)''(x_{i-1})=2/3$ $[2N_{i-1}(h)-N_i(h)]$ $-(10/3)\,f(u_i)+(8/3)\,f(v_i)$

and $(2/81)p^2(s_i^h)''(x_i)=(2/3)[2Ni(h)-N_{i-1}(h)]$ $+(8/3)\,f(u_i)-(10/3)\,f(v_i)$.

Thus,

$(2/27)p^2\left[((s_i^h)''-(s_{i-1}^h)'')(x_{i-1})\right]$

$=(2/3)[N_{i-2}(h)-f_{i-2}]-(N_i(h)-f_i)]+V_i(f)$ (3.12)

where

$V_i(f)=(10/3)[f(v_{i-1})-f(u_i)]+(8/3)[f(v_i)-f(u_{i-1})]$ $+(1/3)[f_{i-2}-f_i]$.

Again using the Taylor's theorem appropriately, we see that,

$$\|V_i(f)\|\le(82/9)\,p^2.w(f'',p) \quad (3.13)$$

and therefore, using (3.6) and (3.13), we have from (3.12)

$$\|((s_i^h)-(s_{i-1})''(x_{i-1})\|\le K_3(h,1/3)\,w(f'',p) \quad (3.14)$$

where $K_3(h,1/3)=[18K_1(h,85/9]$.

Thus, combining (3.9), (3.11) and (3.14), we get

$$\|((s_i^h)-f'')(x)\|\le(1+K_2(h,0)^t+K_3(h,1/3)\,w(f'',p), \quad (3.15)$$

which proves the result of Theorem 3.1 for r=2, with $K(h,1/3)=1+K_2(h,1/3)+K_3(h,1/3)$.

Next, we observe that in view of the interpolatory condition (1.2) and (1.3), there exist a point $t_i\in(u_i,v_i)$ s.t. $(s_i^h)'-f')(t_i)=0$.

Thus, for any $x\in[x_{i-1},x_i]$

$$\|(t^{(1)}-f')(x)\|\le\max_i\left|\int_{Q_i}^{x}(t^{(2)}-f'')(q)dq\right|$$

$$\le\theta^*p\|(t^{(2)}-f'')(q)\|, \quad (3.16)$$

which along with (3.15) gives the result of Theorem 3.1 for r=1.

Since $(t^{(0)}-f)(u_i)=0$

we finally get

$$\|(t^{(0)}-f)(x)\|\le\max_i\left|\int_{u_i}^{x}(t^{(1)}-f')(q)\,d_2\right|$$

$$\le(2/3)\,p\|(t^{(1)}-f')(q)\|. \quad (3.17)$$

This completes the proof of Theorem 3.1.

4. Difference between Two Extrapolated Splines

Considering two values u,v of h, we propose to compare in this section two extrapolated cubic splines in the classes $S(3,P,u)$ and $S(3,P,v)$ which are the interpolant of Theorem 2.1.

In this section, we shall prove the following :

Theorem 4.1. Suppose s^h is 1 periodic spline of Theorem 2.1 interpolating to the periodic function *f*. Then, for $h=u,v>0$,

$$\|(s^u-s^v)(x)\|\le2|v-u|K(1/3,u,v)\|A^{-1}(u)\| \quad (4.1)$$

where $K(1/3,u,v)$ is a positive function which depends on *u and* v.

Proof of Theorem 4.1. For any function g, we define, the operation $\delta_{u,v}$ by $\delta_{u,v}\,g=g(u)-g(v)$ and for convenience, we write δ for $\delta_{u,v}$.

Thus, we see that (2.6) imples

$$(2/9)p^3(s_i^u - s_i^v)(x) = R_i(x)\delta N_i - Q_i(x)A\delta N_{i-1}. \quad (4.2)$$

Rewrite the equation (3.3) for h=u and h=v, respectively, we assume at the following equality.

$$A(u)\delta N = \delta F - N(v)\delta A. \quad (4.3)$$

Further, in view of (3.4), we have

$$\| A^y(u) \| \le \{2u(p-u)p^n\}^{-1}. \quad (4.4)$$

Next we observe that the matrix δ_A has at the most three non-zero elements. Thus, substractly the matrix A(u) from A(v), we see that

$$\| \delta_A \| \le 2|v-u|p^5. \quad (4.5)$$

Also, we observe that

$$\| N(v)\}\} \le \| A^{-1}(v) \|.\| F(v) \|. \quad (4.6)$$

Further, we have

$$\| \delta F \| \le \| v-u \| (p^2 + 3p(v+u) + 6(v^2+u^2+2uv)p^3 w(f,p) \quad (4.7)$$

Thus, combining (4.3) - (4.7), we have

$$\| \delta N \| \le \| v-u \| K(1/3,u,v) \| A^{-1}(u) \|, \quad (4.8)$$

where $K(1/3,u,v)$ is a operator function which depends on u and v.

Finally, in view of (4.2) and observing that

$$\max_i |R_i(x)| \le (3/2)p^3, \quad (4.9)$$

This is, complete the proof of theorem 4.1.

5. Results and Discussion

The existence, uniqueness and convergence properties of extrapolated cubic spline We have take different interpolatory and boundary conditions to construct cubic polynomial and find error bounds for cubic spline.

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