On T – Closed Submodules

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Abstract: ***In this paper, we introduce T – Closed Submodules. Let T, A and B be submodules of a module M. A is called a T – closed submodule of M (denoted by A $\leq_{T\text{-}c}$ M) , whenever A $\leq_{T\text{-}e}$ B then A = B + T .We investigate the basic properties of a T- Closed submodules.***

Keywords: closed submodules , T – essential submodules

1. Introduction

Throughout this paper , rings are associative with identity and modules are unitary left R-modules. Recall that a submodule A of an R-module M is essential (or large)in M, denoted by A $\leq_e$ M , in case for every submodule B of M , A ∩ B = 0 implies B = 0 , see [1] . And Recall that a submodule A of a module M is said to be a closed submodule (briefly A $\leq_c$ M) if A has no proper essential extension of M , that is , if the only solution of the relation A $\leq_e$ B ≤ M is A = B , see [2]. More details about essential submodules and closed submodules can be found in [3]-[4]. Let A be a submodule of the R – module M .A submodule B of a module M is called a complement of A in M if it is maximal in the set of submodules B of M with A ∩ B = 0 , see [5] . In [6], the authors introduced the concept of essential submodules with respect to an arbitrary submodule. Recall that , let T be a proper submodule of a module M . A submodule A of M is called T-essential submodule denoted by A $\leq_{T\text{-}e}$ M provided that A ≰ T and for each submodule B of M, A ∩ B≤ T implies that B ≤ T . And introduced the definition of T – complement , as follows : Let T be a proper submodule of a module M , and let A be a submodule of M . A submodule B of M is called a T – complement to A in M if B is maximal with respect to the property that A ∩ B ≤ T . In section 2 ,we introduce the definition of T- closed submodule as follows : Let T, A and B be submodules of a module M. A is called a T– closed submodule of M denoted by A$\leq_{T\text{-}c}$ M , whenever A $\leq_{T\text{-}e}$ B then B = A + T . And we give some properties about T - closed submodule of a module M ,We show that If A + T ≤ M then M has a T – closed submodule B such that A + T $\leq_{T\text{-}e}$ B ,see proposition 2.12 . In section 3, we have presented more characteristics about T – closed submodules. We prove that If B is T– Complement to A in M then B $\leq_{T\text{-}c}$ M ,see theorem 2.18. Also we prove that , let T, A and N are submodules of a module M such that T ≤ A ∩ N. If A $\leq_{T\text{-}c}$ M and N $\leq_{T\text{-}e}$ M then A ∩ N $\leq_{T\text{-}c}$ N , see Proposition 3.7.

2. The T- closed submodules

In this section we present a variety of characterizations around T - Closed submodules .We start this section by the following definition:

Definition 2.1.Let T,A and B be submodules of a module M. A is called a T–closed submodule of M (denoted by A $\leq_{T\text{-}c}$ M), whenever A$\leq_{T\text{-}e}$ B then B = A + T .

Let M be a module and let T = 0 . For a submodule A of M . Clearly that A is a T – closed in M if and only if A is closed in M.

Examples

1) Consider Z as Z – module . Let K = mZ , T = nZ and Z = mZ + nZ . Claim that mZ $\leq_{nZ\text{-}e}$ H ≤ Z .Since nZ , mZ ≤ Z, then mZ + nZ = Z ≤ H . But H≤ Z , therefore H = Z. Thus mZ $\leq_{nZ\text{-}c}$ Z.
2) Consider Z_6 as Z– module . Let A =$\{\bar{0},\bar{3}\}$ and T = $\{\bar{0}\}$, since A is not$\{\bar{0}\}$ – essential in Z_6 ,then $\{\bar{0},\bar{3}\}$ is $\{\bar{0}\}$ – closed in Z_6. Since$\{\bar{0},\bar{3}\}$ is not essential in Z_6 Then$\{\bar{0},\bar{3}\}$ is closed in Z_6.
3) Consider Z_6 as Z – module . Let A = $\{\bar{0},\bar{3}\}$ and T = $\{\bar{0},\bar{2},\bar{4}\}$, then A $\leq_{T\text{-}e}$ Z_6 and Z_6 = A + T .Thus A is a T– closed in Z_6 . And since $\{\bar{0},\bar{3}\}$ is not essential in Z_6 ,then $\{\bar{0},\bar{3}\}$ is closed in Z_6 .
4) Consider Z_8 as Z – module . Let A = $\{\bar{0},\bar{2},\bar{4},\bar{6}\}$ and T = $\{\bar{0},\bar{4}\}$, then $\{\bar{0},\bar{2},\bar{4},\bar{6}\}$ is $\{\bar{0},\bar{4}\}$ - essential in Z_8 . But A+ T ≠ Z_8 , therefore $\{\bar{0},\bar{2},\bar{4},\bar{6}\}$ is not $\{\bar{0},\bar{4}\}$ - closed in Z_8.
5) Consider Z_{12} as Z – module. Let A = $\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8}\}$ and T = $\{\bar{0},\bar{6}\}$ Since A is not T – essential in Z_{12}. Then A is a T – closed in Z_{12}. But A is essential in Z_{12}, therefore A is not closed in Z_{12}.

Proposition 2.2. [6]

Let T , A and B be submodules of a module M . Then

1) If A $\leq_{T\text{-}e}$ M then (A+ T) / T $\leq_e$ M / T .
2) If T ≤ A then A$\leq_{T\text{-}e}$ M if and only if A / T $\leq_e$ M / T .
3) K$\leq_{T\text{-}e}$ M if and only if for each m∈ M –T, there exist r∈ R such that r m ∈ K− T .
4) If A and B are T – essential submodules of M , then A ∩ B is T – essential too .
5) Let A ≤ B ≤ M and T ≤ B .Then A $\leq_{T\text{-}e}$ M if and only if A $\leq_{T\text{-}e}$ B and B $\leq_{T\text{-}e}$ M .
6) Let f : N → M be a epimorphism . If A $\leq_{T\text{-}e}$ M , then f^{-1} (K) $\leq_{f^{-1}(T)-e}$ N .
7) If T ≤ A then there exists a submodules B of M such that A + B $\leq_{T\text{-}e}$ M and (A + B) / T = (A / T) ⊕ ((B + T) / T)
8) We prove the following

Remark 2.3. Let T and A be submodules of a module M , if there exist a submodule B of M such that T ≤ A ⪇ B and A $\leq_{T\text{-}e}$ B ≤ M then A is not a T – closed in M.

Proof: suppose that A ⪇ B and A $\leq_{T\text{-}e}$ B ≤ M . Assume that A is a T – closed in M. Then A + T = A = B , but A + T ≠ B therefore which is a contradiction. Thus A is not a T – closed in M.

Remark 2.4. Let T and A be submodules of a module M such that $T \leq A$, then $A \leq_{T-c} M$ if and only if whenever $A \leq_{T-e} B \leq M$ then A = B.

Proof: Clear that by definition.

Remark 2.5. For each T and A be submodules of a module M such that $A \leq T$ then $A \leq_{T-c} M$

Proof: Assume that $A \leq T$, then by definition of T – essential submodules , M has not T – essential submodules A of M . Thus $A \leq_{T-c} M$.

Example 2.6: Z as Z – module . Let K = 4Z and T = 2Z , since $4Z \leq 2Z$. Then 4Z is not 2Z – essential in Z .Thus 4Z is 2Z – closed in Z.

Proposition 2.7: Let T and A be submodules of a module M. If $(A + T) / T \leq_c M / T$ then $A \leq_{T-c} M$. **Proof :** Let $A \leq_{T-e} B \leq M$, to show A + T = B . By proposition 2.2-2 , then $(A + T) / T \leq_e B / T \leq M / T$. But $(A + T) / T \leq_c M / T$, therefore (A + T) / T = B / T . Thus A + T = B .

Note: The converse of proposition 2.7, is not true , show that by example

Example 2.8.Consider Z as Z – module . Let T = 12Z and A = 6Z ,(A + T) / T = (6Z + 12 Z) / 12Z = 6Z / 12Z , M / T = Z / 12Z . To show 6Z / 12Z is not closed submodule in Z / 12Z. Then by (6,example 2.9) , $6Z / 12Z \leq_e Z / 12$. Thus 6Z / 12Z is not closed submodule in Z / 12Z. But 6Z is not 12Z – essential of Z, see (6,example 2.9), therefore 6Z is 12Z – closed of Z .

Proposition 2.9: Let T and A be submodules of a module M then $(A + T) / T \leq_c M / T$ if and only if $A + T \leq_{T-c} M$.

Proof: ⇒) Suppose that $A + T \leq_{T-e} B \leq M$. To show A+ T = B. By proposition 2.2-2, then $(A + T) / T \leq_e B / T \leq M /T$. But $(A + T) / T \leq_c M /T$, therefore (A + T) / T = B / T . Thus A + T = B.
⇐) Let $(A + T) / T \leq_e B / T \leq M / T$. To show (A + T) / T = B / T. By proposition 2.2-2, then $A + T \leq_{T-e} B \leq M$. But $A + T \leq_{T-c} M$, therefore A + T = B. Thus (A+ T) / T = B / T .

Corollary 2.10. Let T and A be submodules of a module M such that $T \leq A$ then $A / T \leq_c M / T$ if and only if $A \leq_{T-c} M$.

Proof : Clear by proposition 2.9.

Proposition 2.11. Let T and A be submodules of a module M . If $A \leq_{T-c} M$ and $A \leq_{T-e} A + T \leq M$ then $(A + T) / T \leq_c M / T$.

Proof : - Assume that $A \leq_{T-c} M$ and $A \leq_{T-e} A + T \leq M$, to show $(A+ T) / T \leq_c M / T$. Let $(A + T) / T \leq_e B / T \leq M / T$, to show (A + T) / T = B / T. By proposition 2.2-2, then $A + T \leq_{T-e} B$, and since $A \leq_{T-e} A + T$. By proposition 2.2-5, then $A \leq_{T-e} B$. But $A \leq_{T-c} M$, therefore A + T = B . Hence (A + T) / T = B / T .Thus $(A + T) / T \leq_c M / T$.

Proposition 2.12.If T + A be a submodule of a module M then M has a T– closed submodule B such that $A + T \leq_{T-e} B$.

Proof :- Let $A + T \leq M$ and $F =\{ D \leq M \mid A + T \leq_{T-e} D\}$. Clearly that $A + T \in F$, and hence $F \neq \varphi$. Let $\{ C_\alpha \}_{\alpha \in \Lambda}$ be a chain in F .To show that $\bigcup_{\alpha \in \Lambda} \{ C_\alpha \}$ in F. Clearly $\bigcup \{C_\alpha \}_{\alpha \in \Lambda}$ is a submodule of M , now to show $A + T \leq_{T-e} \bigcup_{\alpha \in \Lambda} \{ C_\alpha \}$. Let $x \in \bigcup_{\alpha \in \Lambda} \{ C_\alpha \} - T$. To show there exist $r \in R$ such that $rx \in (A + T) - T$,let $x \in C_\alpha$, then $x \in C_\alpha - T$. Since $A + T \leq_{T-e} C_\alpha$,then there exist $r \in R$ such that $rx \in (A + T) - T$. Thus $\bigcup_{\alpha \in \Lambda} \{ C_\alpha \} \in F$. By Zorn's lemma F has a maximal element say H , then $A + T \leq_{T-e} H$. Claim that $H \leq_{T-c} M$. Let $H \leq_{T-e} L \leq M$. To show H + T = L . Since $A + T \leq_{T-e} H \leq_{T-e} L$. By proposition 2.2-5 , then $A + T \leq_{T-e} L$, and hence $L \in F$. Which is a contradiction by a maximal element , hence H = L . Thus H + T = L.

Corollary 2.13. Let T and A be submodules of a module M such that $T \leq A$, then M has a T – closed submodule B such that $A \leq_{T-e} B$.

Proof :- clearly by proposition 2 .12.

Theorem 2.14. Let T , A and B be submodules of a module M and $A + T \leq_{T-c} B + T \leq_{T-c} M$ then $A+ T \leq_{T-c} M$.

Proof : - Let $A + T \leq_{T-c} B + T \leq_{T-c} M$, by proposition 2.9, then $(A+ T) / T \leq_c (B + T) / T \leq_c M / T$. [7, proposition.1.5,p.18] $(A + T) / T \leq_c M / T$, by proposition 2.9, then $A + T \leq_{T-c} M$.

Corollary 2.15. Let T, A and B be submodules of a module M such that $T \leq A \cap B$. If $A \leq_{T-c} B \leq_{T-c} M$ then $A \leq_{T-c} M$.

Proof : - clear by theorem 2.14 .

Corollary 2.16. Let T, A and B be submodules of a module M such that $A \leq B \leq M$ and $T \leq B$, if $A \leq_{T-c} M$ then $A \leq_{T-c} B$.

Proof :- Assume that $A \leq_{T-e} N \leq B \leq M$. To show A + T = N , since $A \leq_{T-e} N \leq M$ and $A \leq_{T-c} M$, then A + T = N . Thus $A \leq_{T-c} B$.

Theorem 2.17. Let T and A be submodules of a module M . Then A is a T - Closed in M if and only if for each $B \leq M$ such that $A \leq B + T$ then A is a T - Closed in B + T . **Proof :⇒)** Suppose $A \leq_{T-c} M$, to show $A \leq_{T-c} B + T$. Let $A \leq_{T-e} N \leq B + T$, to show A + T = N . Since $A \leq_{T-e} N \leq B + T \leq M$ and $A \leq_{T-c} M$. Then A + T = N . ⇐) Clear .

Theorem 2.18. Let T , A and B be submodules of a module M. If B is T– Complement to A in M then $B \leq_{T-c} M$. **Proof :** Let $B \leq_{T-e} N \leq M$ .To show B + T = N , since $A \cap B \leq T$, then $B \cap (A \cap N) \leq T$. Since $B \leq_{T-e} N$, then $A \cap N \leq T$. But B is maximal with respect to the property that $A \cap B \leq T$, therefore B = N. Thus B + T = N .

Note. The converse of theorm2.18 is not true for example: Consider Z_{12} as Z –module .Let $A=\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8}\}$, $B=\{\bar{0},\bar{4},\bar{8}\}$ and $T = \{\bar{0},\bar{6}\}$. Since B is not T –essential in Z_{12}, then B is T – closed in Z_{12}. We want to show B is not a T- complement to A in Z_{12} . Since A ∩ B $=\{\bar{0},\bar{4},\bar{8}\} \not\leq$ T , then B is not T-complement to A in Z_{12}.

3. Characterizations of T – Closed Submodules

In this section we give various characterizations. of T-Closed submodules. The following tow theorems gives a characterization of T – closed submodules.

Theorem 3.1. Let A + T be a submodule of a module M. Then the following statement are equivalent:
1- A + T $\leq_{T-c}$ M
2- If A + T ≤ B + T $\leq_{T-e}$ M then B .+ T $\leq_{(A+T)-e}$ M .
3- If B + T is a T – complement to A + T in M then A + T is a T – complement to B + T in M .
4- A + T is a T – complement for some B + T submodule of M .

Proof :- 1⇒2) Assume that A + T $\leq_{T-c}$ M and A + T ≤ B + T $\leq_{T-e}$ M , to show B + T$\leq_{(A+T)-e}$ M . By proposition 2.9, then (A + T) / T $\leq_c$ M / T , and by proposition 2.2-2, then (A + T) / T ≤ (B+ T) / T $\leq_e$ M / T. By [7,proposition 1.4,page 18] [((B + T) / T] / [(A+ T) / T] $\leq_e$ [M / T] / [(A + T) / T]. Then by the third isomorphism theorem [(B + T) / T] / [(A + T) / T] ≅ [(B + T) / (A+ T)] and [M / T] / [(A + T) / T] ≅ [M / (A + T)] .Hence [(B + T) / (A+ T)] $\leq_e$ [M / [(A + T)] .Thus by proposition 2.2-2 , B + T $\leq_{(A+T)-e}$ M .

Proof :- 2⇒3) Let B + T is a T– complement to A + T in M . To show A + T is a T – complement to B + T in M . Let A + T ≤ N ≤ M such that (B + T) ∩ N ≤ T, to show A + T = N . Since B + T is a T – complement to A + T in M , then by proposition2.2-7, then (A + T) + (B + T) $\leq_{T-e}$ M , thus (A+ B + T) $\leq_{T-e}$ M. Since A + T ≤ (A + T) + B $\leq_{T-e}$ M, then by (2) A + B + T $\leq_{(A+T)-e}$ M . Since A ≤ A + T ≤ N , then (A + B + T) ∩ N = ((B + T) ∩ N) + A ≤ T + A by modular law . Since A + B+ T $\leq_{(A+T)-e}$ M , then N ≤ A + T .But A + T ≤ N , therefore A + T = N . Thus A + T is a T – complement to B + T in M .

Proof :- 3⇒4) To show A + T is a T– complement for some B + T submodule of M . Since A + T ≤ M ,then by Zorn's lemma A + T has a T – complement say B + T in M . By (3) A + T is a T – complement to B + T in M .

Proof :- 4⇒1) Let A + T is a T – complement for some B + T submodule of M . To show A + T $\leq_{T-c}$ M . Let A + T $\leq_{T-e}$ N ≤ M, to show A + T = N .Since A + T is a T –complement for some B+ T submodule of M. Then [(A+ T) ∩(B + T ≤ T] ∩ N. Hence [(A + T) ∩ (B + T)] ∩ N ≤ T ∩ N = T. Since A + T $\leq_{T-e}$ N, then (B+ T) ∩ N ≤ T . But B + T is maximal with respect to the property (A + T) ∩ (B + T) ≤ T, therefore A + T = N . Thus A + T $\leq_{T-c}$ M .

Theorem 3.2. Let T and A be submodules of a module M such that T ≤ A. The following statement are equivalent:
1- A is T – closed in M . 2- If A ≤ B + T $\leq_{T-e}$ M then B + T $\leq_{A-e}$ M . 3- If B + T is a T – complement to A in M then A is a T – complement to B + T in M. 4- A is a T – complement for some B + T submodule of M . **Proof :-** 1⇒2) Suppose that A $\leq_{T-c}$ M . By Corollary 2.10, then A / T $\leq_c$ M / T . Since A ≤ B + T $\leq_{T-e}$ M , then by proposition 2.2-2, A / T ≤ (B + T) / T $\leq_e$ M / T, [5, proposition 1.4 ,page18] then ((B + T) / T) / (A / T) $\leq_e$ (M / T) / (A / T). By third isomorphic (B + T) / A $\leq_e$ M / A, then by proposition 2.2-2, then B + T$\leq_{A-e}$ M .

Proof:- 2⇒3) Let B + T is a T–complement to A in M. To show A is a T–complement to B + T in M. Let A ≤ N ≤ M such that (B + T) ∩ N ≤ T, to show A= N. Since B + T is a T – complement to A in M , by proposition 2.2-7, then (B + T) + A $\leq_{T-e}$ M .Since A ≤ ((B+ T) + A)$\leq_{T-e}$ M .then by (2), (B + T) + A $\leq_{A-e}$ M .Since A ≤ N ≤ M, then by modular law ((B + T) + A) ∩ N = ((B + T) ∩ N) + A ≤ T + A = A . Since (B + T) + A $\leq_{A-e}$ M , then N ≤ A. But A ≤ N, therefore N = A. Thus A is a T – complement to B + T in M.

Proof :- 3⇒4) To show A is a T– complement for some B + T submodule of M . Since A ≤ M ,then by Zorn's lemma A has a T – complement say B + T in M . Thus by (3) A is a T – complement to B + T in M .

Proof :- 4⇒1) Let A is a T – complement for some B + T submodule of M . To show A $\leq_{T-c}$ M . Let A $\leq_{T-e}$ N ≤ M , to show A + T = N .Since A is a T – complement to B + T in M , then A ∩ (B + T) ≤ T , hence[A ∩ (B + T)] ∩ N ≤ T ∩ N= T . Implies that A ∩ [(B + T) ∩ N] ≤ T .Since A$\leq_{T-e}$ N, then (B + T) ∩ N ≤ T . But A is maximal with respect to property that (B + T) ∩ A ≤ T , therefore A = N . Thus A + T = N .

Proposition 3.3. Let T, A and N be submodules of a module M. Consider the following statement:
1- A + T $\leq_{T-c}$ N.
2- A + T ≤ B + T $\leq_{T-e}$ N for each N≤ M then B + T $\leq_{(A+T)-e}$ N. Then 1⇒2.

Proof : 1⇒2) Suppose that A + T $\leq_{T-c}$ N ≤ M and A + T ≤ B + T $\leq_{T-e}$ N ,∀ N ≤ M. To show B + T $\leq_{(A+T)-e}$ N. Since A + T $\leq_{T-c}$ N, then by proposition 2.9, (A+ T) / T $\leq_c$ N / T . And since A + T ≤ B + T $\leq_{T-e}$ N , then by proposition 2.2-2, (A + T) / T ≤ (B + T) / T $\leq_e$ N / T. By [7, proposition1.4, page18] then, [(B + T / T) / (A + T / T)] $\leq_e$ [(N / T) / (A + T) / T]. By the third isomorphism theorem [(B+ T / T) / ((A + T) / T)] ≅ [(B + T) / (A+ T)] and [(N / T) / ((A + T) / T)] ≅ [(N / (A + T)] . Hence (B + T) / (A+ T) $\leq_e$ N / (A + T) . Thus by proposition 2.2-2, B + T $\leq_{(A+T)-e}$ N.

Corollary 3 .4. Let T , A and N be submodules of a module M and T ≤ A .
1- A + T $\leq_{T-c}$ N.
2- A + T ≤ B + T $\leq_{T-e}$ N for each N ≤ M then B + T $\leq_{A-e}$ N. Then1⇒2.

Proof: Clear by proposition3.3 .

Proposition 3.5. Let T , A and B are submodules of a module M such that A ≤ B. If B $\leq_{(T+A)-c}$ M . Then B / A $\leq_{[(T+A)/A]-c}$ M / A .

Proof :- Let B / A $\leq_{[(T+A)/A]-e}$ N / A ≤ M / A . To show [(B / A) + ((T + A) / A)] = N / A , implies that (B + T) / A = N / A . Let f : N→ N / A be a natural epimorphism. By proposition 2.2-6 , then $f^{-1}(B / A) \leq_{f^{-1}((T+A)/A)-e} N$. Hence $B \leq_{(T+A)-e} N \leq M$. But $B \leq_{T-c} M$, therefore B + T = N. Thus(B + T) / A = N / A .

Proposition 3.6. Let T and A be submodules of M 1- If $A \leq B + T \leq_{T-e} M$ then $B + T / A \leq_{((A+T)/A)-e} M / A$. 2- If B + T is a T – complement to A + T in M then A + T is a T – complement to B+ T in M.
Then 1⇒2.

Proof :- 1⇒2) Let B + T is a T – complement to A + T in M. To show A + T is a T – complement to B + T in M . Since B + T is a T – complement to A + T in M , then $(B + T) \cap (A+ T) \leq T$. Let $A + T \leq N \leq M$ such that $(B+ T) \cap N \leq T$, to show A+ T = N . Since B + T is a T – complement to A + T in M . By proposition 2.2-7 , then $(B + T)+ (A + T) \leq_{T-e} M$, implies that $(B + A + T) \leq_{T-e} M$. Since $A \leq B + A + T \leq_{T-e} M$, then by (1) $(B + A + T) / A \leq_{((A+T)/A)-e} M / A$. Since $N /A \leq M /A$, then $((B + A + T) / A) \cap (N / A) = ((B + A + T) \cap N) / A = [((B + T)\cap N) + A / A] \leq [(T + A) / A]$. Since $(B + A + T) /A \leq_{((A+T)/A)-e} M / A$. Then $N / A \leq (T + A) / A$, implies that $N \leq T + A$. But $A + T \leq N$, therefore A + T = N .Thus A + T is a T – complement to B + T in M .

Proposition 3.7. Let T, A and N be submodules of a module M such that $T \leq A \cap N$. If $A \leq_{T-c} M$ and $N \leq_{T-e} M$ then $A \cap N \leq_{T-c} N$.

Proof :- Suppose $A \leq_{T-c} M$ and $T \leq A$, then by Corollary 2.9, $A / T \leq_{c} M / T$. Hence by [5, proposition 1.4 ,page18] A / T is a complement to B / T in M / T . Claim that A is a T - complement to B in M .Since A / T is a complement to B / T in M / T, then A / T is maximal with respect to the property $(A / T) \cap (B / T) = 0$. Hence $A \cap B = T$, now let $A \leq N \leq M$ such that $B \cap N \leq T$ .Since $B \cap N \leq T = A \cap B$. But $A \leq N$, therefore $A \cap B \leq B \cap N$. Hence $A \cap B = A \cap N$, then $[(A / T) \cap (B / T)] = (A \cap B) / T = (A \cap N) / T = T / T = 0$. But A / T is a complement for B / T in M / T , therefore N / T = A / T , so N = A .Thus A is a T – complement to B in M . Claim that $A \cap N$ is a T - complement to $B \cap N$ in N . Let $A \cap N \leq L \leq N$ such that $(B \cap N) \cap L \leq T$. To show $A \cap N = L$. Since $N \leq_{T-e} M$, then $B \cap L \leq T$. Claim that $(A + L) \cap B \leq T$. Since $((A + L) \cap B) \cap N = ((A + L) \cap N) \cap B$ and since $L \leq N$, then by modular law $((A \cap N) + L) \cap B = L \cap B \leq T$. Hence $(A + L) \cap B \leq T$. But A is a maximal with respect to the property $B \cap A \leq T$, therefore A + L = A. Hence $L \leq A$, implies that $L = L \cap N \leq A \cap N$. But $A \cap N \leq L$, therefore $A \cap N = L$. Thus by theorem 2.18, $A \cap N \leq_{T-c} N$.

Proposition 3.8. Let T and A be submodules of a module M
1- A is a T – closed in M.
2- If $A \leq B \leq_{(A+T)-e} M$ then $B/A \leq_{[(T+A)/A]-e} M/ A$.
Then 1⇒2.

Proof : Let $A \leq_{T-c} M$ and $A \leq B \leq_{(A+T)-e} M$. To show $B/A \leq_{[(T+A)/A]-e} M/ A$. Let $C /A \leq M/A$ such that $(B/A) \cap (C /A) \leq (T + A) /A$. To show $(C / A) \leq (T + A) /A$.
Since $(B \cap C) / A \leq (T + A) /A$, then $B \cap C \leq T + A$. Since $B \leq_{(A+T)-e} M$. Then $C \leq T + A$, then $(C / A) \leq (T + A) /A$.
Thus $B/A \leq_{[(T+A)/A]-e} M/ A$.

Definition 3.9. Let M be an R – module. Recall that Z (M) = {x ∈ M ; ann (x) $\leq_e$ R } is called the singular submodule of M . If Z (M) = M , then M is called the singular module . If Z (M) = 0 then M is called nonsingular module , [7] .

Proposition 3.10. Let T and A be submodules of a module M and M / (A+ T) is non singular then $A+ T \leq_{T-c} M$.
Proof :- Let $A + T \leq_{T-e} N \leq M$, to show A+ T = N . Since M / (A + T) is non singular and $N / (A + T) \leq M / A+T$, then N / (A + T) is non singular. Since $A + T \leq_{T-e} N$, by proposition 2.2-2, then $(A + T) / T \leq_e N / T$. Hence by [7, page32], (N / T) / ((A + T) / T) is singular . By third isomorphic theorem then(N / T) / ((A + T)/ T) ≅ N/ (A+ T), hence N/ (A+ T) is singular . But the only submodule which is singular and nonsingular is the zero submodule .Therefore N / (A + T) = 0 , Thus A + T = N .

Note. The converse of above proposition is not true . For example: Consider Z_{12} as Z –module . Let N = $\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$ and T = $\{\bar{0},\bar{6}\}$, N + T = N . Since N is not T - essential in Z_{12} , then $\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$ is $\{\bar{0},\bar{6}\}$ - Closed in Z_{12} .And Z_{12} / (N + T) = Z_{12} / $\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$ ≅ Z_2 is singular .

Proposition 3.11. Let T and A be submodules of a module M such that $T \leq A$ and M / T be a non singular R – module . If $A \leq_{T-c} M$ then M / A is non singular.

Proof: We want to show Z (M / A) = 0 , let x + A ∈ M / A with ann (x + A) $\leq_e$ R . To show x + A = A , let $(Rx + A) / A \leq M / A$. Claim that (Rx + A) / A is singular , let w = (r x + a) + A = r x + A . Since ann (x + A) ≤ ann (r x + A) ≤ R ,by [1, proposition5.16, page 74] , then ann (r x + A) $\leq_e$ R , thus (R X + A) / A is singular. Then by third isomorphic theorem, [(R X + A) / A] ≅[((R X + A) / T) / (A / T)], then((R X + A) / T) / (A / T) is singular. By [7, proposition 1.21,page 32] then $A / T \leq_e (R X + A) / T$,by proposition 2.2-2, $A \leq_{T-e} (R X + A)$. Since $A \leq_{T-c} M$. Then A = R X + A. Since x ∈ RX ≤ A ,then x + A = A . Thus M / A is non singular.

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