Stability and Consistency Analysis for Implicit Scheme for MHD Stokes Free Convective Fluid flow Model Equations Past an Infinite Vertical Porous Plate in a Variable Transverse Magnetic Field

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Abstract: ***This research has discussed an analysis of the stability of the implicit Scheme for solving nonlinear partial differential equations used to investigate MHD free convective flow of an incompressible fluid past an infinite vertical porous plate with joule heating in presence of a variable transverse magnetic field .The derivation of the implicit Scheme schemes has been presented. The stability and consistency properties of the implicit scheme are described. Von-Neumann method is used to analyze stability of the implicit scheme where the eigenvalue of the amplification matrices are tested and confirmed to be less than one. The scheme is confirmed to be unconditionally stable. Taylor's series expansion of every term in the scheme is done to analyze consistency of the implicit scheme developed where the original PDE (momentum equation) is recovered from the schemes suggesting that they are consistent. The scheme is found to be unconditionally stable and convergent*.**

Keywords: Momentum Equation, Implicit scheme, Von-Neumann method, Stability and Consistency

1. Introduction

This section begins with discussion of the linear and nonlinear partial differential equations used to investigate MHD Stokes free convective flow of an incompressible fluid past an infinite vertical porous plate with joule heating in presence of a variable transverse magnetic field. A review of Finite Difference Method and numerical schemes developed in our study is also looked at. Towards the end of the paper, the concept of Consistency and Stability analysis of the schemes developed are tested and discussed. Many real life problems generally do not have analytical solutions. Mathematics being one of the scientific research disciplines that lead to real life situations requires numerical techniques to accomplish non-analytical solutions. Doyo and Gofe (2016) considered the convergence rates and stability of the Forward Time Centered Space (FTCS) and Backward Time Centered Space (BTCS) schemes for solving one-dimensional, time-dependent diffusion equation with Neumann boundary condition. It was found that both methods are first order accurate in the spatial dimension. It is shown that An Alternating Direction Explicit Scheme is stable if the modulus of the Eigenvalue of the Amplification matrix should be less than or equal to one. The method is unconditionally stable, since finite difference discretization converges at the rate of the Truncation Error (TE) if the exact solution is smooth enough. The exact solution at the mesh points of the scheme is expanded with a Taylor series and inserted into the scheme developed to calculate the TE (difference between the resulting equation and the original PDE) and determine its order in the approximation used. It is seen that as the discrete step sizes approach to zero, their TE also approaches to zero which indicates that the difference approximations are consistent. Drazin (1996) discusses the stability of the finite difference schemes for solving the nonlinear Klein-Gordon equation. The methods he discusses are those that were developed by Kruskal *et al* (1979).Tinega *et al* (2016) solved the two dimension Sine-Gordon equation used in explaining a number of physical phenomena including the propagation of fluxons in Josephson junctions using Finite Difference Method. An Alternating Direction Implicit numerical scheme for the equation is developed with concepts of stability tested using Matrix Method. The results obtained indicated that the Alternating Direction Implicit numerical scheme is unconditionally stable. The results also indicate that when the surface damping parameter increases, the current flowing through the long Josephson junction also increases. Tinega *et al* (2018) considered the stability of the An Alternating Direction Explicit Scheme for solving two-dimensional Sine-Gordon Equation .The derivation of the An Alternating Direction implicit Scheme schemes was presented and described the stability and consistency of the scheme developed. The scheme was found to be unconditionally stable and convergent. Nyachwaya *et al* (2014) solved third order seepage parabolic partial differential equation (which models the fluid flows) and analyzed stability of the schemes developed by two types of finite differences methods, which are Alternating Direction Explicit (ADE) method and Alternating Direction Implicit (ADI) method subject to some boundary and initial conditions. Numerical stability of both methods by matrix Method was studied. It was observed that both schemes are conditionally stable.

In view of the foregoing pertinent literature presented above, it can be inferred that the problem of Magnetohydrodynamic laminar unsteady flow of an incompressible fluid past an infinite vertical porous plate has received little attention particularly in analyzing the basic properties (i.e Stability and Consistency) of the schemes developed for the governing equations. To quantify how well the finite difference technique performs in generating a solution to a problem, the two fundamental criteria (i.e Stability and Consistency) have to be applied and analysed to compare and contrast the results for different methods outlined in the foregoing literature.

1.2 The Model Equations

The non-dimensionalized general governing equations for MHD free convective flow of an incompressible fluid past an infinite vertical porous plate with joule heating in presence of a variable transverse magnetic field are the momentum and energy equations; (Amenya *et al*, 2013, Sigey *et al*, 2013);

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+Gr\theta+M^2u \quad (1)$$

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\Pr}\left(\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}\right)+E_cM^2u^2 \quad (2)$$

Where Gr, M^2, Pr and E_c are the Grashof number, magnetic parameter, Prandtl number and Eckert numbers respectively.

2. Stability of the Implicit Schemes

Stability considerations are very important in getting the numerical solution of a differential equation using Finite Difference Methods. The stability analysis for the implicit numerical scheme developed for momentum equation is done using either Fourier (Von-Neumann method) or Matrix Methods. In the two methods, the matrix method includes the effect of boundary conditions while the Von-Neumann method excludes the effect of boundary conditions which are used to investigate stability. Both methods are attributed to John von Neumann.

2.1 Discretization of momentum equation

Discretization of momentum equation (1) is only considered in this study. The partial derivatives in momentum equation (1) are replaced by their finite approximations. This discretization gives the scheme

$$\frac{U_{i,j}^{n+1}-U_{i,j}^{n}}{(\Delta t)}+u_{i,j}^{n}\frac{U_{i+1,j}^{n}-U_{i-1,j}^{n}}{2(\Delta x)}+v_{i,j}^{n}\frac{U_{i,j+1}^{n}-U_{i,j-1}^{n}}{2(\Delta y)}=\frac{U_{i+1,j}^{n}-2U_{i,j}^{n}+U_{i-1,j}^{n}}{(\Delta x)^2}+\frac{U_{i,j+1}^{n}-2U_{i,j}^{n}+U_{i,j-1}^{n}}{(\Delta y)^2}+Gr\theta_{i,j}^{n}+M^2u_{i,j}^{n} \quad (3)$$

If we take $Gr=M^2=\theta_{i,j}^{n}=u_{i,j}^{n}=v_{i,j}^{n}=1$, $(\Delta x)=(\Delta y)$, and let $r=\Delta t/(\Delta x)^2=\Delta t/(\Delta y)^2$, $\phi=\Delta t/(\Delta x)=\Delta t/(\Delta y)$ and multiply (3) by $2\Delta t$ throughout, we get implicit scheme

$$(4r+2)U_{i,j}^{n+1}-2rU_{i+1,j}^{n+1}-2rU_{i-1,j}^{n+1}=(2-4r)U_{i,j}^{n}+(2r-\phi)U_{i,j+1}^{n}+(2r+\phi)U_{i,j-1}^{n}-\phi U_{i+1,j}^{n}+\phi U_{i-1,j}^{n} \quad (4)$$

2.2 Von-Neumann method for stability analysis of implicit scheme (4)

The primary observation in the Fourier or Von-Neumann method is that the numerical scheme is linear and therefore it will have a solution in the form $u(x,y,t)=\lambda^t e^{Ix\psi}e^{Iy\psi}$. Thus, a numerical scheme is stable provided that $|\lambda|<1$ and unstable whenever $|\lambda|>1$ Shanthakumar (1989). We can apply this method by substituting the trivial solution in finite difference method at the time t by $u_{i,j}^{n}=\lambda^t e^{Ix\psi}e^{Iy\psi}$ when x,y >0 , $I=\sqrt{(-1)}$; Douglas (1955) and Lapidus and Pinder (1982). We assume $u_{i,j}^{n}=\lambda^t e^{Ix\psi}e^{Iy\psi}$ and Substitute into (4) then, we have

$$\begin{aligned}&(4r+2)\lambda^{t+1}e^{Ix\psi}e^{Iy\psi}-2r\lambda^{t+1}e^{I(x+1)\psi}e^{Iy\psi}-2r\lambda^{t+1}e^{I(x-1)\psi}e^{Iy\psi}=(2-4r)\lambda^t e^{Ix\psi}e^{Iy\psi}\\&+(2r-\phi)\lambda^t e^{Ix\psi}e^{I(y+1)\psi}+(2r+\phi)\lambda^t e^{Ix\psi}e^{I(y-1)\psi}-\phi\lambda^t e^{I(x+1)\psi}e^{Iy\psi}+\phi\lambda^t e^{I(x-1)\psi}e^{Iy\psi}\end{aligned} \quad (5)$$

Dividing (5) by $\lambda^t e^{Ix\psi}$, we get

$$(4r+2)\lambda-2r\lambda e^{I\psi}-2r\lambda e^{-I\psi}=(2-4r)+(2r-\phi)e^{I\psi}+(2r+\phi)e^{-I\psi}-\phi e^{I\psi}+\phi e^{-I\psi} \quad (6)$$

Making λ the subject of the formula (6)

$$\lambda=\frac{(2-4r)+(2r-2\phi)e^{I\psi}+(2\phi+2r)e^{-I\psi}}{(4r+2)-2re^{I\psi}-2re^{-I\psi}} \quad (7)$$

By Eulers formula

$$\left.\begin{aligned}e^{I\psi}&=\cos\psi+i\sin\psi\\e^{-I\psi}&=\cos\psi-i\sin\psi\end{aligned}\right\} \quad (8)$$

Substituting (8) into (7) we get

$$\lambda=\frac{(2-4r)+(2r-2\phi)(\cos\psi+i\sin\psi)+(2\phi+2r)(\cos\psi-i\sin\psi)}{(4r+2)-2r(\cos\psi+i\sin\psi)-2r(\cos\psi-i\sin\psi)} \quad (9)$$

Upon simplification of (9), we get

$$\lambda=\frac{2-4r+4r\cos\psi-i4\phi\sin\psi}{4r+2-4r\cos\psi} \quad (10)$$

where λ is the amplification factor. For stable situation we require that $|\lambda|\le 1$. Separating the real (Re) and imaginary (Im) parts in (10) we get

$$\lambda=\frac{2-4r+4r\cos\psi}{4r+2-4r\cos\psi}-i\frac{4\phi\sin\psi}{4r+2-4r\cos\psi} \quad (11)$$

Taking the real (Re) and imaginary (Im) parts of separately in (11)

$$\text{Re}(\lambda)=\frac{2-4r+4r\cos\psi}{4r+2-4r\cos\psi} \quad (12)$$

And

$$\mathrm{Im}(\lambda) = \frac{-4\phi\sin\psi}{4r+2-4r\cos\psi} \quad (13)$$

For the stability requirement

$$\left|\mathrm{Re}(\lambda)\right| = \left|\frac{2-4r+4r\cos\psi}{4r+2-4r\cos\psi}\right| \le 1 \quad (14)$$

And

$$\left|\mathrm{Im}(\lambda)\right| = \left|\frac{4\phi\sin\psi}{4r+2-4r\cos\psi}\right| \le 1 \quad (15)$$

With λ being the amplification factor which tell whether the error is bounded for determining stability.
We determine stability for the largest value of the amplification factor λ. For the largest value of λ we take $\psi = 90^o$, Substituting into (14) and (15) ,we get

$$\left|\mathrm{Re}(\lambda)\right| = \left|\frac{2}{4r+2}\right| = \left|\frac{1}{2r+1}\right| \le 1 \quad (16)$$

And

$$\left|\mathrm{Im}(\lambda)\right| = \left|\frac{-4\phi}{4r+2}\right| = \left|\frac{-2\phi}{2r+1}\right| \le 1 \quad (17)$$

Obviously $|\lambda|$ will always be less than 1 for the equations (16) and (17). The above cases are always satisfied as the left inequality of Equations (16) and (17) requires. Thus the Implicit difference scheme (4) is stable for all values of $r, \phi > 0$, i.e conditionally stable.

3. Consistency Analysis

Consistency requires that the original equation can be recovered from the algebraic Equations. Obviously this is a minimum requirement for any discretization. In the following it is illustrated how this can be done in terms of a Taylor expansion of the discretized of momentum equation for implicit scheme (4) developed.

3.1 Consistency of the Momentum Equation

We expand every individual term of the equation (4) using Taylors series expansion

$$U_{i+1,j}^{n+1} = U_{i,j}^{n} + \left(hu_x + hu_t\right) + \frac{1}{2!}\left(h^2u_{xx} + 2h^2u_{xt} + h^2u_{tt}\right) + \frac{1}{3!}\left(h^3u_{xxx} + 3h^3u_{xxt} + 3h^3u_{xtt} + h^3u_{ttt}\right) \quad (18)$$

$$U_{i-1,j}^{n+1} = U_{i,j}^{n} + \left(-hu_x + hu_t\right) + \frac{1}{2!}\left(h^2u_{xx} - 2h^2u_{xt} + h^2u_{tt}\right) + \frac{1}{3!}\left(-h^3u_{xxx} + 3h^3u_{xxt} - 3h^3u_{xtt} + h^3u_{ttt}\right) \quad (19)$$

$$U_{i,j}^{n+1} = [U_{i,j}^{n} + hu_t + \frac{h^2}{2!}u_{tt} + \frac{h^3}{3!}u_{ttt} + \frac{h^4}{4!}u_{tttt} + \ldots.]\ldots \quad (20)$$

$$U_{i,j+1}^{n} = U_{i,j}^{n} + hu_y + \frac{h}{2!}u_{yy} + \frac{h^3}{3!}u_{yyy} + \frac{h^4}{4!}u_{yyyy} + \ldots \quad (21)$$

$$U_{i,j-1}^{n} = U_{i,j}^{n} - hu_y + \frac{h^2}{2!}u_{yy} - \frac{h^3}{3!}u_{yyy} + \frac{h^4}{4!}u_{yyyy} + \ldots\ldots \quad (22)$$

$$U^{n}{}_{i+1,j} = U^{n}{}_{i,j} + \Delta x u_x + \frac{(\Delta x)^2}{2!}u_{xx} + \frac{(\Delta x)^3}{3!}u_{xxx} + \frac{(\Delta x)^4}{4!}u_{xxxx} + \ldots\ldots.. \quad (23)$$

$$U^{n}{}_{i-1,j} = U^{n}{}_{i,j} - \Delta x u_x + \frac{(\Delta x)^2}{2!}u_{xx} - \frac{(\Delta x)^3}{3!}u_{xxx} + \frac{(\Delta x)^4}{4!}u_{xxxx} + \ldots\ldots.. \quad (24)$$

Substituting (18), (19), (20) , (21), (22), (23) and (24) into implicit scheme (4), simplifying and collecting like terms together

$$2\Delta t u_t + 2\phi\Delta x u_x + 2\phi\Delta y u_y = -2r\Delta x^2 u_{xx} + 2r\Delta y^2 u_{yy} - 2r\Delta t^2 u_{tt} + \frac{2}{3}\Delta x^3 u_{xxx} \quad (25)$$

If we take $(\Delta x) = (\Delta y)$, and let $r = \Delta t/(\Delta x)^2 = \Delta t/(\Delta y)^2$, $\phi = \Delta t/(\Delta x) = \Delta t/(\Delta y)$ so that (25) becomes

$$2\Delta t u_t + 2\left(\frac{\Delta t}{\Delta x}\right)\Delta x u_x + 2\left(\frac{\Delta t}{\Delta y}\right)\Delta y u_y = -2\left(\frac{\Delta t}{\Delta x^2}\right)\Delta x^2 u_{xx} + 2\left(\frac{\Delta t}{\Delta y^2}\right)\Delta y^2 u_{yy} - 2\left(\frac{\Delta t}{\Delta x^2}\right)\Delta t^2 u_{tt} + \frac{2}{3}\Delta x^3 u_{xxx} \quad (26)$$

Simplifying (26), we get

$$2\Delta t u_t + 2\Delta t u_x + 2\Delta t u_y = -2\Delta t u_{xx} + 2\Delta t u_{yy} - 2\left(\frac{\Delta t}{\Delta x^2}\right)\Delta t^2 u_{tt} + \frac{2}{3}\Delta x^3 u_{xxx} \quad (27)$$

Dividing (26) by $2\Delta t$ throughout, we get

$$u_t + u_x + u_y = u_{xx} + u_{yy} - \frac{1}{\Delta x^2} u_{tt} + \frac{1}{6\Delta t} \Delta x u_{xxx}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{\Delta x^2} u_{tt} + \frac{1}{6\Delta t} \Delta x u_{xxx}$$

(28)

Where the error is $E^n_{i,j} = -\frac{1}{\Delta x^2} u_{tt} + \frac{1}{6\Delta t} \Delta x u_{xxx}$

It is noted that the first five terms in equation (28) are for the recovered PDE that is (momentum equation) and the other terms is the truncation error, since the momentum equation has been recovered from the algebraic equation of the implicit scheme developed, we therefore conclude that the scheme is consistent with the momentum PDE.

4. Conclusion

We have analysed the stability and consistency of implicit scheme for the governing equations for MHD Stokes free convective flow of an incompressible fluid past an infinite vertical porous plate with joule heating in presence of a variable transverse magnetic field. The implicit scheme is found to be unconditionally stable and convergent.

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Nomeclature

M^2	Magnetic Parameter
Gr	Grashof number
Pr	Prandtl number
IFDS	Implicit Finite Difference Scheme
PDE	Partial Differential Equation
MHD	Magnetohydrodynamics
FDM	Finite Difference Method
TE	Truncation Error

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