MINIMUM DEGREE ENERGY FOR SOME CLASSES OF GRAPHS

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Abstract

In this paper we find minimum degree energy for graphs like $C_n \times P_2$, Totalgraph of C_n, Totalgraph of regular graph of degree r and Cocktail party graph

Keywords and Phrases : Minimum degree matrix, minimum degree eigenvalues, minimum degree energy of a graph.

2010 AMS Subject Classification : 05C30, 05C50.

1 Introduction

Let G be a simple graph and let its vertex set be $V(G) = \{v_1, v_2, ..., v_n\}$. The adjacency matrix $A(G)$ of the graph G is a square matrix of order n whose (i, j)-entry is equal to unity if the vertices v_i and v_j are adjacent, and is equal to zero otherwise. The eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of $A(G)$, assumed in non increasing order, are the eigenvalues of the graph G.

The energy of G was first defined by I.Gutman[4] in 1978 as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Ever since the graph energy $E(G)$ of a simple graph G was introduced by I.Gutman [4], there is a constant stream of papers devoted to this topic. Survey of development before 2001 can be found in [5]. For recent developments one can consult [3]. The energy of a graph has close links to chemistry(see for instance [6] . Let G be a simple graph with n vertices $v_1, v_2, \ldots, v_n$ and let d_i be the degree of v_i, $i = 1, 2, 3, \ldots, n$. Define

$$d_{ij} = \begin{cases} min\{d_i, d_j\} & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the $n \times n$ matrix $m(G) = (d_{ij})$ is called the minimum degree matrix of G. This was introduced and studied in[1, 2]. The characteristic polynomial of the minimum degree matrix $m(G)$ is defined by

$$\phi(G; \mu) = det(\mu I - m(G))$$
$$= \mu^n + c_1 \mu^{n-1} + c_2 \mu^{n-2} + \ldots + c_{n-1}\mu + c_n,$$

where I is the unit matrix of order n. The minimum degree eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of the graph G, assumed in the non increasing order, are the eigenvalues of its minimum degree matrix $m(G)$. The minimum degree energy of a graph G is defined as

$$E_m(G) = \sum_{i=1}^{n} |\mu_i|.$$

Since $m(G)$ is real symmetric matrix with zero trace, these minimum degree eigenvalues are all real with sum equal to zero. In this paper, we compute the minimum degree Energy $E_m(G)$ of some classes of graphs like $C_n \times P_2$, Totalgraph of C_n,

Totalgraph of regular graph of degree r and Cocktail party graph.

2   MIMINIMUM DEGREE ENERGY

In this section we find minimum degree energy of some classes of graphs.

Definition 2.1   If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs then the cartesian product of G_1 and G_2 denoted by $G_1 \times G_2 = G(V, E)$ consists a vertex set $V = V_1 \times V_2$ and $(x_1, y_1), (x_2, y_2) \in V_1 \times V_2$ are adjacent if $x_1 x_2 \in E_1$ and $y_1 = y_2$ or $y_1 y_2 \in E_2$ and $x_1 = x_2$.

Theorem 2.1 *The minimum degree energy of $C_n \times P_2$ is*

$$E_m(C_n \times P_2) = \sum_{i=1}^{n} \left| -1 + 4\cos\frac{2\pi i}{n} \right| + \sum_{i=1}^{n} \left| 1 + 4\cos\frac{2\pi i}{n} \right|.$$

Proof. The minimum degree characteristic polynomial of C_n is

$$P_{C_n}(\mu) = -2^{n+1} + 2^n \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \left(\frac{\mu}{2} \right)^{n-2k}.$$

The minimum degree eigenvalues of cycle C_n are the numbers $4\cos\frac{2\pi i}{n}$, for $i = 1, 2, 3, \ldots, n$. The minimum degree characteristic polynomial of P_2 is

$$P_{P_2}(\mu) = \mu^2 - 1,$$

the minimum degree eigenvalues os P_2 are -1 and 1. The eigenvalues of $C_n \times P_2$ are $-1 + 4\cos\frac{2\pi i}{n}$ and $1 + 4\cos\frac{2\pi i}{n}$ $(i = 1, 2, 3, \ldots, n)$. Thus

$$E_m(C_n \times P_2) = \sum_{i=1}^{n} \left| -1 + 4\cos\frac{2\pi i}{n} \right| + \sum_{i=1}^{n} \left| 1 + 4\cos\frac{2\pi i}{n} \right|.$$

Also the minimum degree energy of $C_n \times P_3$ is

$$E_m(C_n \times P_3) = \sum_{i=1}^{n} \left| 4\cos\frac{2\pi i}{n} \right| + \sum_{i=1}^{n} \left| -2 + 4\cos\frac{2\pi i}{n} \right| + \sum_{i=1}^{n} \left| 2 + 4\cos\frac{2\pi i}{n} \right|.$$

Definition 2.2   The total graph $T(G)$ of graph G is that graph whose set of vertices is the union of the set of vertices and of the set of edges of G, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident.

Let G be regular graph of degree r, having n vertices and m edges, then the minimum degree matrix $m(T(G))$ of total graph $T(G)$ with vertex set $(v_1, v_2, \ldots, v_n, e_{n+1}, e_{n+1}, \ldots, e_{m+n})$ can be written in the form,

$$m(T(G)) = \begin{pmatrix} A & R \\ R^T & B \end{pmatrix},$$

where $A = [a_{ij}]$,

$$a_{ij} = \begin{cases} 2r & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise,} \end{cases}$$

$B = [b_{ij}]$,

$$b_{ij} = \begin{cases} 2r & \text{if } e_{n+1} \text{ and } e_{n+j} \text{ are adjacent} \\ 0 & \text{otherwise,} \end{cases}$$

and $R = [r_{ij}]$,

$$r_{ij} = \begin{cases} 2r & \text{if } v_i \text{ and } e_{n+j} \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Hence the characteristic polynomial $\phi(T(G); \mu)$ of $m(T(G))$ is obtained as follows,

$$P_{T(G)}(\mu) = (2r)^{m+n} \begin{vmatrix} \frac{\mu}{2r}I + rI - RR^T & -R \\ -R^T & \frac{\mu}{2r}I + 2I - R^TR \end{vmatrix}.$$

where $\rho = \frac{\mu}{2r}$.

$$P_{T(G)}(\mu) = (2r)^{m+n} \begin{vmatrix} (\rho+r)I - RR^T + \frac{R}{\rho+2}\left(-(\rho+r+1)R^T + R^TRR^T\right) & 0 \\ -(\rho+r+1)R^T + R^TRR^T & (\rho+2)I \end{vmatrix}.$$

$$P_{T(G)}(\mu) = (2r)^{m+n}(\rho+2)^m \left|\rho I - m(G) + \frac{1}{\rho+2}(m(G)+rI)(m(G)-(\rho+1)I)\right|$$

$$= (2r)^{m+n}(\rho+2)^{m-n} \left|(m(G))^2 - (2\rho - r + 3)m(G) + (\rho^2 - (r-2)\rho - r)I\right|$$

$$= (2r)^{m+n}(\rho+2)^{m-n} \prod_{i=1}^{n} \left(\rho_i^2 - (2\rho - r + 3)\rho_i + \rho^2 - (r-2)\rho - r\right)$$

$$= (2r)^{m+n}(\rho+2)^{m-n} \prod_{i=1}^{n} \left(\rho^2 - (2\rho_i + r - 2)\rho + \rho_i^2 + (r-3)\rho_i - r\right),$$

where $\rho_i, (i = 1, 2, 3, \ldots, n)$ are minimum degree eigenvalues of A. Now we prove the following theorems.

Theorem 2.2 *If G is a regular graph of degree r, with n vertices and m edges, then*

$$E_m(T(G)) = |4r(m-n)| + n \sum_{i=1}^{n} \left|\left(2r\rho_i + r^2 - 2r + r\sqrt{r^2 + 4 + 4\rho_i}\right)\right|$$

$$+ n \sum_{i=1}^{n} \left|\left(2r\rho_i + r^2 - 2r - r\sqrt{r^2 + 4 + 4\rho_i}\right)\right|.$$

Proof. The minimum degree characteristic polynomial of $T(G)$ is

$$\phi(T(G); \mu) = (2r)^{m+n}\left(\frac{\mu}{2r} + 2\right)^{m-n} \prod_{i=1}^{n} \left(\frac{\mu^2}{4r^2} - (2\rho_i + r - 2)\frac{\mu}{2r} + \rho_i^2 + (r-3)\rho_i - r\right).$$

Hence minimum degree eigenvalues of $T(G)$ are $-4r$ [$(m-n)$ times] and $2\rho_i r + r^2 - 2r \pm r\sqrt{r^2 + 4 + 4\rho_i}$ [n times].

Thus

$$E_m(T(G)) = |4r(m-n)| + n \sum_{i=1}^{n} \left|\left(2r\rho_i + r^2 - 2r + r\sqrt{r^2 + 4 + 4\rho_i}\right)\right|$$

$$+n \sum_{i=1}^{n} \left| \left(2r\rho_i + r^2 - 2r - r\sqrt{r^2 + 4 + 4\rho_i} \right) \right|.$$

Theorem 2.3 *For all $n \geq 3$, the minimum degree of $T(C_n)$ is ,*

$$E_m(T(C_n)) = |8(m-n)| + n \sum_{i=1}^{n} \left| \left(16 \cos \frac{2\pi i}{n} + 4\sqrt{2 + 4\cos \frac{2\pi i}{n}} \right) \right|$$

$$+n \sum_{i=1}^{n} \left| \left(16 \cos \frac{2\pi i}{n} - 4\sqrt{2 + 4\cos \frac{2\pi i}{n}} \right) \right|.$$

Proof. Since C_n is a regular graph of degree 2, the minimum degree characteristic polynomial of $T(C_n)$ is

$$\phi(T(C_n); \mu) = (4)^{m+n} \left(\frac{\mu}{4} + 2 \right)^{m-n} \prod_{i=1}^{n} \left(\frac{\mu^2}{16} - 2\cos \frac{2\pi i}{n} \mu + 16 \cos^2 \frac{2\pi i}{n} - 4\cos \frac{2\pi i}{n} - 2 \right).$$

Hence the minimum degree eigenvalues of $T(C_n)$ is -8 [$(m-n)$ times] and $16 \cos \frac{2\pi i}{n} \pm 4\sqrt{2 + 4\cos \frac{2\pi i}{n}}$ [n times]. Thus

$$E_m(T(C_n)) = |8(m-n)| + n \sum_{i=1}^{n} \left| \left(16 \cos \frac{2\pi i}{n} + 4\sqrt{2 + 4\cos \frac{2\pi i}{n}} \right) \right|$$

$$+n \sum_{i=1}^{n} \left| \left(16 \cos \frac{2\pi i}{n} - 4\sqrt{2 + 4\cos \frac{2\pi i}{n}} \right) \right|.$$

Theorem 2.4 *If G is a cocktail party graph with $n = 2k$ vertices, then*

$$E_m(G) = 2(n-2)^2.$$

Proof. Since G is a cocktail party graph with $n = 2k$, it is regular graph of degree $n - 2$.

Then minimum degree characteristic polynomial of G is

$$\phi(G; \mu) = (\mu - 2(k-1)(2k-2))\mu^k (\mu + 2(2k-2))^{k-1}.$$

Therefore, the minimum degree eigenvalues of G are $2(k-1)(2k-2), 0, -2(2k-2)$ with multiplicity 1, k and $(k-1)$. Hence

$$E_m(G) = |2(k-1)(2k-2)| + |0|k + |-2(2k-2)|(k-1)$$

$$= 4(k-1)(2k-2)$$

$$= 2(n-2)^2.$$

Theorem 2.5 *For $n \geq 2$, the minimum degree energy of the crown S_n^0 is equal to*

$$E_m(S_n^0) = 4(n-1)^2.$$

Proof. The minimum degree characteristic polynomial of S_n^0 is

$$\phi(S_n^0 : \mu) = (\mu^2 - (n-1)^2)^{n-1}(\mu^2 - (n-1)^2),$$

hence minimum degree eigenvalues of S_n^0 are $(n-1)((n-1)times)$, $-(n-1)((n-1)times)$, $(n-1)^2(1time)$ and $-(n-1)^2(1time)$, thus, minimum degree energy of S_n^0 is

$$E_m(S_n^0) = |n-1|\,(n-1) + |-(n-1)|\,(n-1) + \left|-(n-1)^2\right| + \left|(n-1)^2\right|$$

$$E_m(S_n^0) = 4(n-1)^2.$$

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