International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2018): 7.426

Application of Numerical Methods in Transient Analysis

Otoo Henry[1], Buabeng Albert[2], Inkoom Justice[3]

[1, 2, 3]University of Mines and Technology, P. O. Box 237, Tarkwa, Ghana

Abstract: ***Transient analysis of an RLC circuit (or LCR circuit) comprising of a resistor, an inductor, and a capacitor are analysed using the Heun's and the Runge-Kutta 4th order methods. Kirchhoff's voltage and current laws were used to generate equations for voltages and currents across the elements in an RLC circuit. From Kirchhoff's law, the resulting second order differential equations were later transformed into first order differential equations by substitution. The Heun's and Runge-Kutta 4th order methods were then used together with MATLAB simulations to check how changes in resistance affects transient. Errors associated with selected numerical methods were then measured with Big "O" notation (truncation). From the study, it was observed that, the computational values of the Heun's method converged faster than that of Runge-Kutta 4th order method. However, the errors incurred in Runge-Kutta 4th order method were very minimal as compared to that of Heun's method, thus, Runge-Kutta 4th order method was concluded to be more accurate than Heun's method.***

Keywords: RLC Circuit, Numerical Methods, Big "O" Notation, MATLAB Simulations

1. Introduction

Transient is the sudden bursts of energy in an electrical circuit which may damage certain components of the circuit. Transient normally results in changing the state of the components of an electrical circuit. It is very difficult for the capacitor voltage and the inductor current in an electrical circuit to assume a new steady state value. Therefore, transient analysis can be used in determining how the capacitor voltage and the inductor current evolve with time [3]. Transient analysis can be described as the analysis of a system in an unsteady state. If the variables defining the state of a system does not vary with respect to time, the system is said to be in a steady state. If not, then it is in an unsteady state. Transient analysis is very important since it can be used in analysing the performance of any electrical circuit [1]. Thus, for an electrical current or voltage flowing through an electrical circuit, there can be various forms of the voltage or current. For instance, considering circuits which are time-varying signals with resistive circuits, the resulting Kirchhoff's Current Law (KCL) and Kirchhoff's Voltage Law (KVL) are normally in the form of differential equations rather than algebraic equations. But these differential equations are not easily solved analytically when the order is high and complex. Also, for an RLC circuit which is an electrical circuit consisting of a resistor, an inductor and capacitor which are connected either in series or parallel, the circuit equations are integro-differential equations. These equations are converted to ordinary differential equations by differentiating with respect to time. Therefore, analysis of the transient in an RLC circuits can be approached numerically [1],[2].

Numerical methods are one of the best techniques in solving almost all mathematical equations that cannot be solved analytically. A number of these methods have been designed to solve such equations especially, in differential equations [3]. The use of a particular numerical methods depends on its efficiency. Efficiency in numerical methods depends on the stability, cost in terms of time, suitability and accuracy [3]. Of all the mentioned requirements, accuracy plays a wonderful role in the choice of a particular numerical procedure [2]. Without the use of the most accurate method, one might not be able to get accurate solution and this might affect further decisions based on the results.

Therefore, in this paper, two numerical techniques namely; the Runge-Kutta Method and the Heun's Method are used in analysing the transient behaviors in an RLC circuit taken into consideration the damping factor and the change in voltage with respect to time in order to determine which one is the best in terms of accuracy and convergence.

2. Materials and Methods Used

2.1 Heun's Method (Improved Euler's Method)

In computational science and mathematics, Heun's method is referred to as improved Euler's method or second order Runge-Kutta method. It is a numerical procedure for solving ordinary differential equations with a given initial value. This method is regarded as extensions of the Euler's method into two-stage second order Runge-Kutta methods [4].

Euler's method is a method devised from the area under a curve and its modification lead to the improved Euler's method or the Heun's method [5]. The Heun's method results in solving initial value problems more accurately than Euler's method. The improved Euler's formula (Heun's Method) is expressed in Equations (1) and (2).

$$y_{n+1}^{p} = y_n + hf(t_n, y_n) \tag{1}$$

$$y_{n+1}^{c} = y_n + \frac{h}{2}\left\{ f(t_n, y_n) + f(t_{n+1} + y_{n+1}^{p}) \right\} \tag{2}$$

where h is the step function.

Equation (1) represents a predictor equation which gives the immediate value whereas Equation (2) represents a corrector equation which also gives the final approximation of the next integration point. The predictor formula gives an inaccurate result y_{n+1}^{p} which is made more accurate by the corrector equation.

2.2 Runge-Kutta Methods

Runge-Kutta methods are a family of implicit and explicit iterative methods in numerical analysis, which include the routine called the Euler Method, used in temporal discretisation for the approximate solutions of ordinary differential equations. Unlike the Heun's method, there are Runge-Kutta methods of different orders. These methods are derived using the Taylor's series expansion shown in Equation (3).

$$y_{n+1} = y_n + hy_n' + \frac{1}{2}h^2 y_n'' + \cdots + \frac{1}{p}h^p y_n^{(p)} + O(h^{p+1}) \quad (3)$$

The Euler's method which is written in the form of Equation (4):

$$y_{n+1} = y_n + hf(x_n, y_n) \quad (4)$$

is the first order Runge-Kutta procedure.

Runge-Kutta method is an effective method of solving first order ordinary differential equations. A given ordinary differential equation of higher order can be converted to a first order differential equations by substitution. The most widely known member of the Runge–Kutta family of methods is referred to as "RK4" (which is the 4th order Runge-Kutta method), "classical Runge-Kutta method" or commonly known as "the Runge–Kutta method" [6].
The adaptive procedure for the fourth order Runge-Kutta Method according to [1] is given by Equation (5)

$$y_{n+1} = y_n + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6} \quad (5)$$

where $n = 0, 1, 2, \dots$

$$k_1 = hf(t_n, y_n) \quad (6)$$

$$k_2 = hf\left(\frac{t_{n+h}}{2}, y_n + \frac{k_1}{2}\right) \quad (7)$$

$$k_3 = hf\left(\frac{t_{n+h}}{2}, y_n + \frac{k_2}{2}\right) \quad (8)$$

$$k_4 = hf(t_{n+h}, y_n + k_3) \quad (9)$$

where k_1 is the increment based on the slope at the beginning of the interval, using y_n;

k_2 is the increment based on the slope at the midpoint of the interval, using $y_n + (k_1/2)$;

k_3 represents the increment based on the slope at the midpoint, but now using $y_n + (k_2/2)$;

k_4 is the increment based on the slope at the end of the interval, using $y_n + k_3$.

2.3 RLC Circuit

Figure 1 is an RLC circuit made up of a capacitor, an inductor and resistor. This circuit forms a harmonic oscillator for the current which dies away with time with the presence of the resistor. That is, the presence of the resistance reduces the resonant frequency [2].

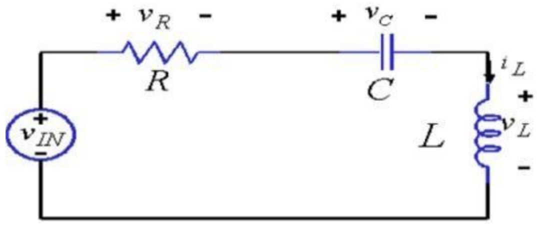

Figure 1: RLC Circuit Diagram

The RLC filter is normally seen as a second-order circuit, meaning that any voltage or current in the circuit can be described by a second-order differential equation in circuit analysis.
Therefore, let

$$I(t) = C\frac{\partial V_c(t)}{\partial t} \quad (10)$$

where C is the capacitance and $V_c(t)$ is the voltage across capacitance. Then, the KVL equation for the circuit is expressed in Equation (11) as:

$$L\frac{\partial I(t)}{\partial t} + RI(t) + V_c(t) = V_{in} \quad (11)$$

where V_{in} is the input voltage.
Substituting $I(t)$ in Equation (11), results in Equation (12)

$$LC\frac{\partial^2 V_c(t)}{\partial t^2} + RC\frac{\partial^2 V_c(t)}{\partial t^2} + V_c(t) = V_{in} \quad (12)$$

Again, for a given RLC circuit, parameters such as, α and ω_0 are written in the form of Equations (13) and (14) respectively.

$$\alpha = \frac{R}{2L} \quad (13)$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (14)$$

where ω_0 is the natural frequency.
Moreover, there is a useful parameter in an RLC circuit, called the damping factor. This damping factor, ζ, is estimated using Equation (15).

$$\zeta = \frac{\alpha}{\omega_0} \quad (15)$$

Suppose the given RLC circuit is in series then the damping factor is estimated using Equation (16).

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} \quad (16)$$

The transient response or type that a circuit exhibits is dependent on the value of the damping factor. The damping factor is the amount by which the oscillation of a system gradually decreases with time [1]. The following are various characterisation of the damping factor:

i. If $\zeta > 1$, the system is called over damped.
ii. If $\zeta = 1$, the system is called critically damped.
iii. If $\zeta < 1$, the system is called under damped.

Since the RLC circuit is described as a second order differential equation, the voltage across 2nd order RLC circuit according to [2] is given by Equation (12). But

$$C\frac{\partial V_c(t)}{\partial t} = I(t) \tag{18}$$

Thus, substituting Equation (18) into Equation (12) yields

$$L\frac{\partial I(t)}{\partial t} + RI(t) + V_c(t) = V_{in}$$

$$\frac{\partial I(t)}{\partial t} = \frac{V_{in} - RI(t) - V_c(t)}{L} \tag{19}$$

Now letting, $I(t) = x_1$ and $V_c(t) = x_2$ leads to Equation (20).

$$\frac{\partial I(t)}{\partial t} = \frac{V_{in} - R(x_1 + x_2)}{L} = g(t, x_1, x_2) \tag{20}$$

$$\frac{x_1}{C} = \frac{\partial Vc(t)}{\partial t} = f(t, x_1, x_2) \tag{21}$$

Formulating Runge-Kutta 4th Order Method for the RLC circuit;

$$\left.\begin{aligned} f_1 &= h * f(t, x_1, x_2) \\ g_1 &= h * g(t, x_1, x_2) \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} f_2 &= hf\left[\left(t + \frac{h}{2}\right), \left(x_1 + \frac{f_1}{2}\right), \left(x_2 + \frac{g_1}{2}\right)\right] \\ g_2 &= hg\left[\left(t + \frac{h}{2}\right), \left(x_1 + \frac{f_1}{2}\right), \left(x_2 + \frac{g_1}{2}\right)\right] \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} f_3 &= hf\left[\left(t + \frac{h}{2}\right), \left(x_1 + \frac{f_2}{2}\right), \left(x_2 + \frac{g_2}{2}\right)\right] \\ g_3 &= hg\left[\left(t + \frac{h}{2}\right), \left(x_1 + \frac{f_2}{2}\right), \left(x_2 + \frac{g_2}{2}\right)\right] \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} f_4 &= hf\left[(t + h), (x_1 + f_3), (x_2 + g_3)\right] \\ g_4 &= hg\left[(t + h), (x_1 + f_3), (x_2 + g_3)\right] \end{aligned}\right\} \tag{25}$$

where h is the step size.

Also, in formulating Heun's method;

$$\left.\begin{aligned} y_{i+1} &= y_i + \frac{h}{2}(m_1 + m_2) \\ z_{i+1} &= z_i + \frac{h}{2}(n_1 + n_2) \end{aligned}\right\} \tag{26}$$

where,

$$\left.\begin{aligned} m_1 &= f(t_i, x_{1i}, x_{2i}) \\ n_1 &= g(t_i, x_{1i}, x_{2i}) \end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} m_2 &= f(t_i + h, x_{1i} + m_1 h, x_{2i} + n_1 h) \\ n_2 &= g(t_i + h, x_{1i} + m_1 h, x_{2i} + n_1 h) \end{aligned}\right\} \tag{28}$$

3. Results and Discussion

3.1 Transient Analysis

To compare the two numerical methods in terms of the transient analysis, the following numerical examples are used.

3.1.1 Numerical Example 1

Taking time to be from zero (0) to 0.001, Let $R = 210\Omega$, $L = 10\,\text{mH}$, $C = 1\text{uF}$ and $V_{in} = 10\,\text{V}$ and substituting the given values in Equations (13), (14) and (15) yields

$$\alpha = \frac{210}{2 * 0.01}$$
$$= 10500$$

$$\omega_0 = \frac{1}{\sqrt{0.01 * 0.000001}}$$
$$= 10000$$

$$\zeta = \frac{10500}{10000}$$
$$= 1.05$$

Hence, the system is over-damped since the damping factor is greater than one (1).

Table 1 shows the voltage outcomes for an overdamped system for both Runge-Kutta 4th order (RK4) method and Heun's method as time increases. From the Table 1, it can be observed that, as time increases, voltage also increases with both numerical methods. It can also be observed that voltage obtained from Heun's method at each time t, is greater than that of Runge-Kutta 4th order method.

Figure 1 shows a plot of voltage against time for both Runge-Kutta 4th order (RK4) and Heun's method. From the Figure 1, it is observed that the Heun's method converges faster than that of Runge-Kutta 4th order method.

Table 1: Results for an Over-Damped System

Time *(t)*	RK4 (Voltage)	Heun's (Voltage)
0.0000	0.000000	0.000000
0.0001	2.920833	5.000000
0.0002	5.933846	7.525000
0.0003	7.849549	8.790125
0.0004	8.907360	9.417963
0.0005	9.456536	9.725861
0.0006	9.732881	9.874598
0.0007	9.869598	9.945043
0.0008	9.936591	9.977520
0.0009	9.969238	9.991922
0.0010	9.985097	9.997930

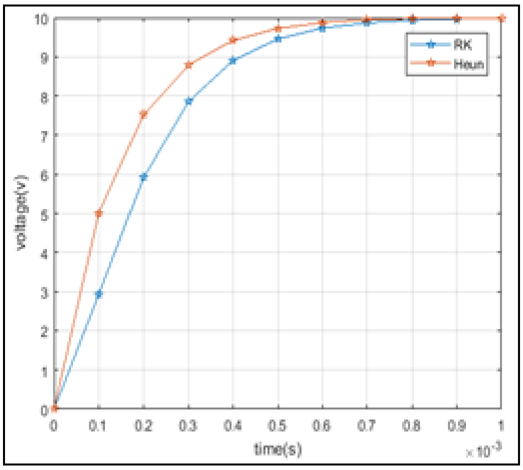

Figure 1: Voltage vs Time Graph for (Overdamped)

3.1.2 Numerical Example 2

Given another system with the same values for C, L and V_{in}. Taking $R = 200\Omega$. Substituting these values into

Equations (13), (14) and (15) yields $\zeta = 1.0$. The value signifies that the system is critically-damped.

Table 2 shows the voltage outcomes for a critically-damped system for both Runge-Kutta 4th order method and Heun's method as time increases. From the Table 2, it can be observed that, as time increases, voltage also increases with both numerical methods. It can also be observed that voltage obtained from Heun's method at each time t, is not always greater than that of Runge-Kutta 4th order method. At some point in time Runge-Kutta values are greater than that of Heun's values.

Table 2: Results for Critically-Damped System

Time *(t)*	RK4 (voltage)	Heun's (voltage)
0.0000	0.000000	0.000000
0.0001	2.916667	5.000000
0.0002	6.093750	7.500000
0.0003	8.066406	8.750000
0.0004	9.099121	9.375000
0.0005	9.596252	9.687500
0.0006	9.823875	9.843750
0.0007	9.924684	9.921875
0.0008	9.968280	9.960938
0.0009	9.986802	9.980469
0.0010	9.994562	9.990234

Figure 2 shows a plot of voltage against time for both Runge-Kutta 4th order and Heun's method. From the Figure 2, it is observed that the plot of Heun's method converges almost at the same time with of Runge-Kutta 4th order method.

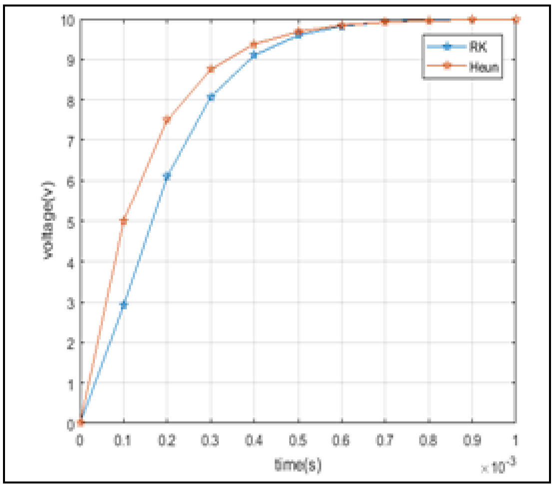

Figure 2: Voltage vs Time Graph for Critically- Damped System

3.1.2 Numerical Example 3

Given another system with the same values for C, L and V_{in}. Taking $R = 100\Omega$. Substituting these values into Equations (13), (14) and (15) yields $\zeta = 0.5$.

From the resulting damping factor value ζ, the system is under-damped.

Table 3 shows the voltage outcomes for an underdamped system for both Runge-Kutta 4th order method and Heun's method as time increases. From the Table 3, it can be observed that, as time increases, voltage also increases in both numerical methods. It can also be observed that voltage obtained from Heun's method at each time t, is not always greater than that of Runge-Kutta 4th order method. At some point in time Runge-Kutta values are greater than that of Heun's values.

Table 3: Results for an Under-Damped System

Time *(t)*	RK4 (voltage)	Heun's (voltage)
0.0000	0.000000	0.000000
0.0001	3.333333	5.000000
0.0002	8.489583	10.000000
0.0003	11.315828	11.250000
0.0004	11.610725	10.625000
0.0005	10.779438	10.000000
0.0006	10.010237	9.843750
0.0007	9.714462	9.921875
0.0008	9.770092	10.000000
0.0009	9.925562	10.019531
0.0010	10.027685	10.009766

Figure 3 shows a plot of voltage against time for both Runge-Kutta 4th order and Heun's method. From the Figure 3, it is observed that the plot of Heun's method converges faster than that of Runge-Kutta 4th order method.

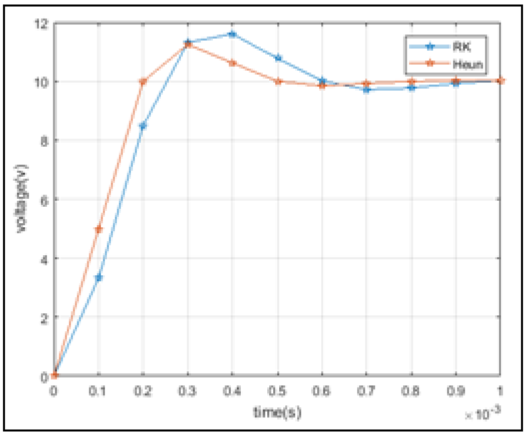

Figure 3: Voltage vs Time Graph for Under-Damped System

3.2 Accuracy Checking

In order to determine the perfect numerical method amongst the two selected techniques with regard to transient analysis, it suffices to check out the errors involve. In this case, the big "O" notation which is also known as the local truncation error was used. The results are shown in Table 4. From Table 4, the local truncation errors indicate that the Runge-Kutta 4th order method is more accurate (1×10^{-20}) than the Heun's method.

Table 4: Errors in Selected Numerical Methods

Error	RK4	Heun's
Minimum	0	0
Maximum	1 x 10^{-20}	1 x 10^{-12}

4. Conclusions and Recommendation

From the study, the Heun's method reaches the stable limit first, thus, converges faster as shown in the Figures 1, 2 and 3. Also, the Runge-Kutta 4th order method proved to be more accurate numerical method for solving higher order differential equations when compared to the Heun's method. Thus, the Runge-Kutta 4th order method is recommended for transient analysis of complex electrical circuits since it is more accurate than the Heun's method.

References

[1] T. A. Kee, and R. Ranom, "Comparison of Numerical Techniques in Solving Transient Analysis of Electrical Circuits", Journal of Engineering and Applied Sciences, Vol. 13, No. 1, pp. 314-320, 2018.
[2] A. Suhag, "Transient Analysis of Electrical Circuits using Runge-Kutta Method and its Application", International Journal of Scientific and Research Publications, Vol. 3, No. 11, pp. 1-5, 2013
[3] A. Adu, "Numerical Method for Solving Differential Equations", Unpublished Bsc Thesis Work, University of Mines and Technology, Tarkwa, Ghana, 2010, 34 pp.
[4] E. Süli, and D. Mayers, An Introduction to Numerical Analysis, Cambridge University Press, 2003, 444 pp.
[5] Z. Jin-Zhou, S. Cui-Cui, L. Yong-Ming and L. Shun-Chu "A Mathematical Model for the Analysis of the Pressure Transient Response of Fluid Flow in Fractal Reservoir", Journal of Chemistry, pp. 1-8, 2015
[6] Ali Filiz "Numerical Method for a Linear Volterra Integro-differential Equation with Cash-Karp Method", Asian Journal of Fuzzy and Applied Mathematics , Vol. 2, 2014, pp1-11

Author Profile

Henry Otoo received the PhD in Mathematics from the University of Mines and Technology (UMaT), Tarkwa, Ghana. and MPhil and BSc degrees in Mathematics from the Kwame Nkrumah University of Science and Technology (KNUST), Kumasi, Ghana. He is a Lecturer at the Department of Mathematical Sciences in the University of Mines and Technology (UMaT), Tarkwa, Ghana.

Albert Buabeng received the MPhil and BSc degrees in Mathematics from the University of Mines and Technology (UMaT), Tarkwa, Ghana. He is a an assistant Lecturer at the Department of Mathematical Sciences in the University of Mines and Technology (UMaT), Tarkwa, Ghana.

Justice Inkoom received the BSc degree in Mathematics from the University of Mines and Technology (UMaT), Tarkwa, Ghana. He is a teaching and research assistant at the Department of Mathematical Sciences in the University of Mines and Technology (UMaT), Tarkwa, Ghana