International Journal of Science and Research (IJSR)
ISSN: 2319-7064
ResearchGate Impact Factor (2018): 0.28 | SJIF (2018): 7.426

On the Extended Generalized Leibnitz Rule for Differentiability of Distributions

Anslem Uche Amaonyeiro

Department of Mathematics, Statistics, Computer Science, Federal University of Agriculture, Makurdi, Nigeria

Abstract: ***In this paper, we present an extended form of partial differentiability of pointwise multiplication of a C^∞ function with a generalized functions (distributions) tions) with compact support to the form of partial derivative of tensor product of C^∞ function with distributions. We extend the result of Leibnitz rule for pointwise multiplication.***

Keywords: Distributions, tensor product, convolution

1. Preliminaries

Definition. Let $\mathcal{E}(\mathbb{R}^n)$ be the space of infinitely differentiable functions on $\mathbb{R}^n$, equipped with the topology of uniform convergence of every derivative on every compact subset of $\mathbb{R}^n$.

We adopt the following notations: $p = (p_1, p_2, \cdots, p_n)$,p_i positive or zero integers, will be a multi - index of differentiation of order $|p| = p_1 + p_2 + \cdots + p_n$, so that, for $\phi \in \mathcal{E}(\mathbb{R}^n)$.

We shall also introduce the convension $p! = p_1!\,p_2! \cdots p_n!$, and that, for $x \in \mathbb{R}^n$, $x = (x_1, x_2, x_3, \cdots, x_n)$, x^p will be $x_1^{p_1}, x_2^{p_2}, \cdots, x_n^{p_n}$,, then the usual Taylor formula is written in $\mathbb{R}^n$ as it in $\mathbb{R}$:

$$\phi(x) = \sum_{p=1}^{\infty} \frac{\phi^{P}(0)}{p!} x^p \qquad (1.1)$$

Among the multi - indices, we introduce an order relation by putting $p \le q$ if $p_1 \le q_1, p_2 \le q_2, \cdots, p_{n \le q_n}$, and an addition, with $p + q = (p_1 + q_1, p_2 + q_2, \cdots, p_n + q_n)$.

The Leintitz formula for the deriavtive of a product, is written as

$$(uv)^{(p)} = \sum_{q \le p}^{\infty} \frac{p!}{q!(p-q)!} u^{(q)} v^{(p-q)} \qquad (1.2)$$

The topology of $\mathcal{E}(\mathbb{R}^n)$ may be defined by the family of seminorms $P_{m,k}$:

$$P_{m,k}(\phi) = \max |\phi^{(p)}(x)|$$

Definition

The support of a continuous function φ is the smallest closed subset of $\mathbb{R}^n$ outside which φ is zero or varnishes or complement of the closure of a set. $\mathcal{D}_K(\mathbb{R}^n)$will be the subspace of $\mathcal{E}(\mathbb{R}^n)$ formed by the functions having their support in the compact subset K of $\mathbb{R}^n$; it will be endowed with the topology induced by $\mathcal{E}(\mathbb{R}^n)$.

Definition

The space $\mathcal{D}(\mathbb{R}^n)$ of infinitely differentiable functions with compact support is the union of the $\mathcal{D}_K$ (Rn), K compact.

Definition

Let $f(x)$ and $g(x)$ be locally integrable functions in the spaces $\mathbb{R}^m$ and $\mathbb{R}^m$ respectively. That is, if $\int_{\mathfrak{R}}^{\cdot} |f(x)|dx < \infty$ and $\int_{\mathfrak{R}'}^{\cdot} |g(x)|dx < \infty$ over regions $\mathfrak{R}$ and $\mathfrak{R}'$ in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

From the above established definition, the function $f(x)g(x)$ is also locally integrable in $\mathbb{R}^{m+n}$ for all $x \in \mathbb{R}^m, y \in \mathbb{R}^n$. It defines the regular distribution by the given formula

$\langle f(x)g(y)\ ,\ \varphi(x,y)\rangle = \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} g(y)\varphi(x,y)dydx = \langle f(x)\ ,\ \langle g(y)\ ,\ \varphi(x,y)\rangle\rangle$

or

$\langle g(y)f(x)\ ,\ \varphi(x,y)\rangle = \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} f(x)\varphi(x,y)dxdy = \langle g(y)\ ,\ \langle f(x)\ ,\ \varphi(x,y)\rangle\rangle$

This implies that

$\langle f(x)g(y), \varphi(x,y)\rangle = \langle f(x), \langle g(y), \varphi(x,y\ =$

$\langle g(y), \langle f(x), \varphi(x,y)$ for all$\varphi(x,y) \in \mathcal{D}^{m+n} \cong \mathcal{D}(\mathbb{R}^m \times \mathbb{R}^n)$.

Definition

Let the open subsets be $\Omega_1 \subset \mathbb{R}^m$and $\Omega_2 \subset \mathbb{R}^n$. Then the product of the open subsets $\Omega_1 \times \Omega_2$ is given as $\Omega_1 \times \Omega_2 = \{(x,y) : x \in \Omega_1, y \in \Omega_2\} \subset \mathbb{R}^{m+n}$. The product $\Omega_1 \times \Omega_2$ is also an open subset in $\mathbb{R}^{m+n}$.

Definition

Let f be a function on the open subset Ω_1 and g acting on the open subset Ω_2. Then we define the direct or the tensor product of f and g on $\Omega_1 \times \Omega_2$ (given as $(f \otimes g)$: $\Omega_1 \times \Omega_2 \subset \mathbb{R}^{m+n} \mapsto \mathbb{C}$) by

$$(f \otimes g)(x,y) = f(x)g(y)$$

Clearly, it is commutative, that is, $(f \otimes g)(x,y) = (g \otimes f)(y,x)$ for every pair of real numbers$(x,y) \in \Omega_1 \times \Omega_2$.

Remark

Since f and g are polynomials, we can have pointwise multiplication: $(x)g(x) = g(y)f(x)$.

Definition

We define the expression $\mu(x) \otimes t(y)$, the direct product of the distributions$\mu(x) \in \mathcal{D}'(\mathbb{R}^m)$ and $t(y) \in \mathcal{D}'(\mathbb{R}^n)$ for every $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ as the following mathematical expression below:

$$\langle \mu(x) \otimes t(y), \varphi(x,y)\rangle = \langle \mu(x), \langle t(y), \varphi(x,y$$

Definition

The derivative of a distribution is defined in such a way that if this distribution is a usual C^1 function, its derivative coincides with the usual derivative of this function. One is therefore led to put:

$$\langle \mu^{(p)}, \varphi\rangle = (-1)^{|p|}\langle \mu, \varphi^{(p)}\rangle$$

Every distribution (in particular, every locally integrable function) is infinitely differentiable in the sense of distributions. The computation of the derivative is much more complicated than that in classical analysis: it involves different kinds of integral formulae, such as Stokes, Green, and so on.

For instance, the function $\frac{1}{|x|^{n-2}}$ (where $|x| = \sqrt{x_1^2 + +x_n^2}$) is known to be harmonic in the complement of the origin; it is not regular in the neighbourhood of the origin. The classical compu - tation of its laplacian gives the simple answer $\triangle \frac{1}{|x|^{n-2}} = 0$in C_0. But $\frac{1}{|x|^{n-2}}$ is locally integrable in the whole of $\mathbb{R}^n$, and therefore is a distribution over $\mathbb{R}^n$; this distribution must have a laplacian over $\mathbb{R}^n$ and this is

$$\triangle \left(\frac{1}{|x|^{n-2}}\right) = -(n-2)S_n\delta,$$

whereS_n is the area of the unit sphere in $\mathbb{R}^n$.

Definition. Multiplication of two distributions cannot be defined. One can define easily the product of a distribution μ by a function φ of $\mathcal{E}$ in such a way that, if μ is a locally integrable function f, we just get the usual product φf. One puts

$\langle \varphi\mu, \phi\rangle = \langle \mu, \varphi\phi\rangle, \mu \in \mathcal{D}', \phi \in \mathcal{D}, \varphi \in \mathcal{E}$ (1.3).

This multiplication has the usual properties, in particular bilinearity, associativity with multiplication in $\mathcal{E}$ (that is $(\varphi\beta)\mu = \varphi(\beta\mu)$), and Leibnitz's formula for a derivative of a product

$$(\mu T)^{(p)} = \sum_{q \le p}^{\infty} \frac{p!}{q!\,(p-q)!}\mu^{(q)}T^{(p-q)}$$

Many people have tried to extend this very restricted multiplication, but without any great success of course, formula (1.3) holds if φ is only m times continuously differentiable, $\varphi \in \mathcal{E}^m$, provided T is only a distribution of order $\le m$ (here $\mathcal{D}^m$ is the subspace of $\mathcal{E}^m$ formed by the functions of compact support; it is equipped with the inductive limit of the topologies of the space $\mathcal{D}_K^m$; and a distribution μ is of order $\le m$if it can be extended as a continuous linear form on $\mathcal{D}^m$). But it can be easily proved that no product can be defined for two arbitrary distributions so that it possesses reasonable properties (as associativity, Leibnitz formula). It appears more and more that some of the greatest mathematical difficulties in theoretical physics, for instance, in quantum field theory, proceed precisely from this impossibility of multiplication.

Now, we present the main result.

Theorem

Let $\mu \in \mathcal{E}'(\mathbb{R}^n)$ and $\in \mathcal{D}'(\mathbb{R}^n)$. For for every $k = (k_1, k_2, \cdots, k_n) \in \mathbb{N}$, we have

$$D^k(\mu * t) = \sum_{j\le k}^{\infty} \frac{k!}{(k-j)!j!}(D^j\mu) * (D^{k-j}t) \text{in} \mathcal{D}'(\mathbb{R}^n)$$

Similarly

$$D^k(\mu \otimes t) = \sum_{j=0}^{k} \binom{k}{j} (D^j\mu) \otimes (D^{k-j}t) \text{in} \mathcal{D}'(\mathbb{R}^n)$$

Proof: For any $\in \mathcal{D}(\mathbb{R}^n)$, from definition we have

$$\langle D^k(\mu * t), \varphi\rangle = (-1)^{|k|}\langle \mu * t, D^k\varphi\rangle$$
$$= (-1)^{|k|}\langle \mu(x) \otimes t(y), D^k\varphi(x+y)\rangle$$
$$= \langle \mu(x), \langle t(y), (x+y)^k\varphi(x+y)\rangle\rangle$$

But from the binomial expansion of $(x+y)^k = \sum_{j=0}^{k}\binom{k}{j}\ x^j y^{k-j}$, then we have

$$\langle D^k(\mu * t)\ ,\ \varphi\rangle = \langle \mu(x)\ ,\ \langle t(y)\ ,\ \sum_{j=0}^{k}\binom{k}{j} x^j y^{k-j}\varphi(x+y)\rangle\rangle$$

$$= \langle \mu(x), \langle \sum_{j=0}^{k}\binom{k}{j} x^j t(y), y^{k-j}\varphi(x+y)\rangle\rangle$$
$$= \langle \mu(x), \sum_{j=0}^{k}\binom{k}{j} x^j \langle t(y), y^{k-j}\varphi(x+y)\rangle\rangle$$
$$= \langle \mu(x), \sum_{j=0}^{k}\binom{k}{j} x^j \langle y^{k-j}t(y), \varphi(x+y)\rangle\rangle$$
$$= \langle \sum_{j=0}^{k}\binom{k}{j} x^j\mu(x), \langle y^{k-j}t(y), \varphi(x+y)\rangle\rangle$$
$$= \langle \sum_{j=0}^{k}\binom{k}{j} x^j\mu(x) \otimes y^{k-j}t(y), \varphi(x,y)\rangle$$
$$= \langle \sum_{j=0}^{k}\binom{k}{j} x^j\mu(x) * x^{k-j}t, \varphi(x,y)\rangle$$

Also, similarly we have that

$$\langle D^k(\mu \otimes t), \varphi(x,y)\rangle = (-1)^{|k|}\langle \mu \otimes t, D^k\varphi(x,y)\rangle$$
$$= (-1)^{|k|}\langle \mu(x)t(y), D^k\varphi(x,y)\rangle$$
$$= (-1)^{|k|}\langle \mu(x), \langle t(y), D^k\varphi(x+y)\rangle\rangle$$

$$= \langle \mu(x), \langle t(y), (x+y)^k \varphi(x+y) \rangle\rangle$$

$$= \langle \mu(x), \langle t(y), \sum_{j=0}^{k} \frac{k!}{(k-j)!\,j!} x^j y^{k-j} \varphi(x+y) \rangle\rangle$$

$$= \langle \mu(x), \langle \sum_{j\le k}^{k} \frac{k!}{(k-j)!\,j!} x^j t(y), y^{k-j} \varphi(x+y) \rangle\rangle$$

$$= \langle \mu(x), \sum_{j\le k}^{k} \frac{k!}{(k-j)!\,j!} x^j \langle y^{k-j} t(y), \varphi(x+y) \rangle\rangle$$

$$= \langle \sum_{j\le k}^{k} \frac{k!}{(k-j)!\,j!} x^j \mu(x), \langle y^{k-j} t(y), \varphi(x+y) \rangle\rangle$$

$$= \langle \sum_{j\le k}^{k} \frac{k!}{(k-j)!\,j!} x^j \mu(x) \otimes y^{k-j} t(y), \varphi(x,y) \rangle$$

$$= \langle \sum_{j\le k}^{k} \frac{k!}{(k-j)!\,j!} (D^j \mu) \otimes (D^{k-j} t), \varphi(x,y) \rangle$$

Hence

$$D^k(\mu \otimes t) = \sum_{j=0}^{k} \frac{k!}{(k-j)!\,j!} (D^j \mu) \otimes (D^{k-j} t) \ .$$

References

[1] B.P. Damyanov, Multiplication of Schwartz Distributions and Colombeau Generalized Functions. Journal of Applied Analysis, Vol. 5. No. 2 (1999 ,pp 249-260.
[2] D. S., The Theory of Generalized Functions. Second ed., Cambridge Univ. Press, 1982.
[3] B. Fisher, the Product of Distributions, Quart. J. Math. Oxford Ser. (2) 43 (1971).