Complicated Behavior of a Novel 3-D Hénon Map

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Abstract: ***The aim of this paper is to present and analyze a novel 3-D Hénon map, this map exhibiting complicated chaotic behavior. Our proposed map can display new types of chaotic attractors in some bifurcation parameters via quasi-periodic bifurcation route to chaos.***

Keywords: Fixed point, chaotic attractors, 3-D Hénon map.

1. Introduction

In literature [2], the 3-D Hénon map is quadratic map with constant Jacobian matrix determinant, and its inverse map is quadratic, and the coordinates are not decoupled by the action of the map. Several researchers have defined and studied quadratic 3-D chaotic maps such as with quadratic inverse and constant Jacobi [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. It was shown in [1] that the general 3-D quadratic map are classified according to their number of nonlinearities, and for each class, also several examples are given. The second case of one nonlinearity is given by:

$$f(x,y,z) = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x + a_2 y + a_3 z + a_4 y^2 \\ b_0 + b_1 x + b_2 y + b_3 z \\ c_0 + c_1 x + c_2 y + c_3 z \end{pmatrix} \quad (1)$$

In this paper, we present the simplest three-dimensional quadratic map of one nonlinearity $(x,y,z) = H(x,y,z) \in \mathbb{R}^3$ defined by:

$$H(x,y,z) = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 + x - ay^2 \\ x - bz \\ y \end{pmatrix} \quad (2)$$

(2)

Where $(x,y,z) \in \mathbb{R}^3$ and $(a,b) \in \mathbb{R}^2$ are bifurcations parameters. Indeed, the map (2) is quadratic, and has constant Jacobian matrix determinant equal to b, and its inverse map is quadratic, and the coordinates are not decoupled by the action of the map (2). Therefore, it is call the 3-D Hénon map. In this paper we have proposed a new Hénon chaotic map with nonlinearities. The dynamical behaviors of the map (2) are investigated here by both theoretical analysis and numerical simulation.

2. Qualitative analysis of the map

2.1 Fixed points and their stability

When $a > 0$, the fixed points of the Hénon map (2) are S_i ((1+b) y_i ,y_i,y_i), $i = 1,2$, where $y_1 = \frac{1}{\sqrt{a}}$ and $y_2 = -\frac{1}{\sqrt{a}}$. Furthermore, for the Hénon map (2) if $a < 0$, then every orbits of the Hénon map (2) are unbounded, while if $a > 0$, then the Hénon map (2) has possible bounded orbits and possibly chaotic. On the other hand, the Jacobian matrix of the Hénon map (2) evaluated at the fixed points S_i is

$$J_{S_i} = \begin{pmatrix} 1 & -2ay_i & 0 \\ 1 & 0 & -b \\ 0 & 1 & 0 \end{pmatrix} \quad (3)$$

The characteristic polynomial of the matrix J_{S_i} is $P_{S_i}(\lambda) = \lambda^3 - \lambda^2 + (b + 2ay_i)\lambda - b$, then the characteristic polynomial of the matrix J_{S_1} is $P_{S_1}(\lambda) = \lambda^3 - \lambda^2 + (b + 2\sqrt{a})\lambda - b$. According to the Schur-Cohn-Jury criterion [13], we conclude that the fixed point S_1 of the map (2) is asymptotically stable if the following conditions hold: (i) $|b| < 1 < 1 - c$, (ii) $2\sqrt{a} > 0$, and (iii) $1 + b + \sqrt{a} > 0$, (iv) $1 - b^2 > 2\sqrt{a}$. From (ii) we have $a > 0$ and from the conditions (i), (iii) and (iv) we have $, -\sqrt{(1-2\sqrt{a})} < b < \sqrt{(1-2\sqrt{a})}$. Finally, we conclude that the region Ω of asymptotic stability Ω of the fixed point S_1 of the Hénon map (2) is given by:

$$\left\{(a,b): 0 < a \le \frac{1}{4}, -\sqrt{(1-2\sqrt{a})} < b < \sqrt{(1-2\sqrt{a})}\right\} \quad (4)$$

Analogously, the Schur-Cohn-Jury criterion states that the fixed point S_2 of the Hénon map (2) is unstable.

For example, we fix parameter $a = 0.0001, b = -0.25$, the Hénon map (2) has two fixed points $S_1(75,100,100)$ and $S_2(-75,-100,-100)$. The Jacobian matrix of the Hénon map (2) evaluated at S_1 has three eigenvalues: $\lambda_1 = -0.493348, \lambda_2 = 0.521327$, and $\lambda_3 = 0.972021$, we have $|\lambda_{1,2,3}| < 1$, therefore S_1 is stable. The Lyapunov exponents of the Hénon map (2) are $L_1 = -0.70654, L_2 = -0.651377$, and $L_3 = -0.0283777$. And the Jacobian matrix of the Hénon map (2) evaluated at S_2 has three eigenvalues:

$\lambda_1 = -0.506681, \lambda_2 = 0.4811$, and $\lambda_3 = 1.02558$, we have $|\lambda_{1,2}| < 1$, and $|\lambda_3| > 1$, hence S_2 is unstable (S_2 is a saddle). The Lyapunov exponents of the Hénon map (2) are $L_1 = 0.0252597, L_2 = -0.679873$ and $L_3 = -0.731681$.

2.2 A region of chaotic attractors

If at least one fixed point that is not asymptotically stable, i.e., it must be a saddle or unstable, generally indicate chaos in the Hénon map (2).

Let us define the following subsets of R:

$$\begin{cases} \overline{\Omega_1} : a > \frac{1}{4} \\ \overline{\Omega_2} : b \leq -\sqrt{(1-2\sqrt{a})} \\ \overline{\Omega_3} : b \geq \sqrt{(1-2\sqrt{a})} \end{cases} \quad (5)$$

Thus one has:

$$\overline{\Omega} = \cup_{j=1}^{j=3} \overline{\Omega_j}. \quad (6)$$

Here, the subsets Ω are the complements in $\mathbb{R}^2$ of the set Ω given in (4). As a result, we have the following Theorem:

Theorem: If $(a,b) \in \overline{\Omega} = \cup_{j=1}^{j=3} \overline{\Omega_j} \in \mathbb{R}^2$, then the Hénon map (2) has possible chaotic attractors, where $\overline{\Omega}$ is the complements in $\mathbb{R}^2$ of the set Ω.

For example a chaotic attractor for the case with $a = 0.44$ and $b = 0.3$ is shown in Fig.1 (f). On the other hand it is necessary to verify the hyperchaoticity of the attractor by calculating three Lyapunov exponents of the Hénon map (2) using the formula: $L_1 + L_2 + L_3 = ln|det(J)|$ where $det(J) = b = 0.3$ and $L_1 = L_2 = -0.868836$, $L_3 = 0.5337$, hence the map (2) is no hyperchaos since the sum of the Lyapunov exponents is not positive. Finally the attractors and chaotic attractors are investigated numerically in Fig.1 and Fig.2 from the Hénon map (2).

3. Bifurcation analysis of the map

From the above analysis, it is visible that the stability of the fixed point of the Hénon map (2) will be changed along with the change of the parameters of the Hénon map (2), and the Hénon map (2) will also be in different state. The dynamical behaviors of the Hénon map (2) are investigated numerically, then, to determine the long-time behavior and chaotic regions, we numerically computed the bifurcation diagram Fig.3 and Lyapunov exponent Fig.4. The diagram Fig.4.1 exhibit a quasi-periodic bifurcation route to chaos for the selected values of the bifurcation parameter a.

We fix parameter $b = 0.3$, and the initial condition $x = y = z = 0.01$, and let a vary. Then for $a \in [0.04, 0.5]$, the map (2) exhibits the following dynamical behaviors as shown in Fig.3 and Fig.4: For $0.04 \leq a < 0.2102$ the Hénon map (2) converges to fixed point, for values of a in the range $0.2102 \leq a < 0.3758$ the Hénon map (2) converges to a periodic-1 orbit as shown in Fig.1(a), for $0.3758 \leq a < 0.4264$ except for a number of periodic window the Hénon map (2) converges to a quasi-periodic-2 orbit as shown in Fig. 1(b), for $0.4264 \leq a < 0.5$ except for a number of periodic window, the Hénon map (2) converges to a chaotic attractor similar to the one in Fig. 1(c-d).

On the other hand, we fix parameter $a = 0.4$, let b vary. Then for $b \in [0.04, 0.5]$, the Hénon map (2) exhibits the following dynamical behaviors as shown in Fig.5 and Fig.6. For $0.04 \leq b < 0.2378$ except for a periodic-6 windows, the Hénon map (2), converges to a quasi-periodic-1 attractor as shown in Fig.2(a), for $0.2378 \leq b < 0.339$ the map (2) converges to a quasi-periodic-2 attractor, for $0.339 \leq b < 0.3574$ the Hénon map (2) converges to a quasi-periodic-4 orbits as shown in Fig.2(b), for $0.3574 \leq b < 0.5$ except for a number of periodic window (three) the map (2) converges to a chaotic attractor similar to the in Fig.2(c)-(d).

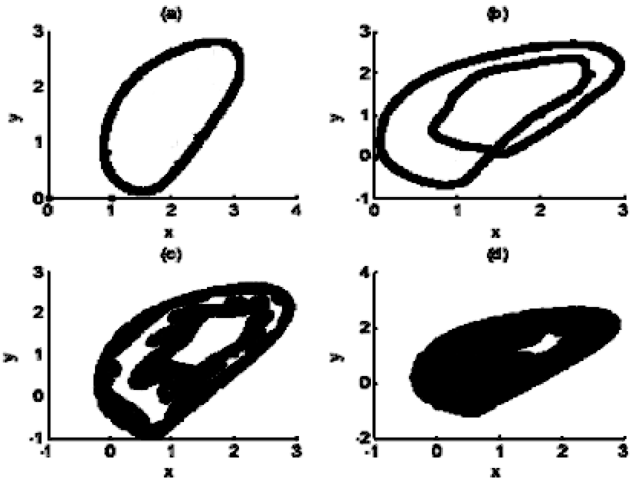

Figure 1: Evolution of Attractors obtained from the Hénon map (2) with $b = 0.3$: and (a) periodic orbit for $a = 0.30$, (b) 2-quasi-periodic orbit, for $a = 0.39$, (c) chaotic attractor for $a = 0.42$, (d) chaotic attractor for $a = 0.44$.

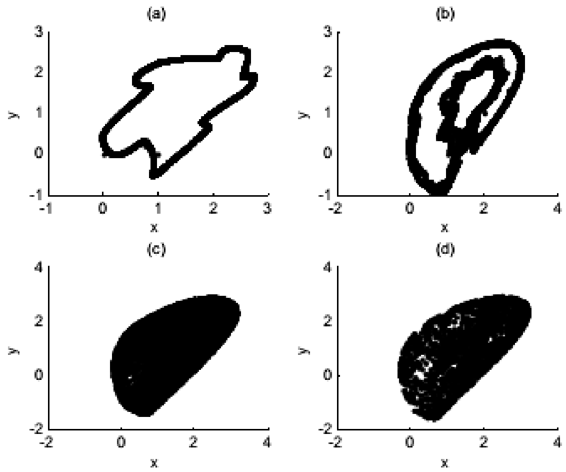

Figure 2: Evolution of Attractors obtained from the Hénon map (2) with $a = 0.4$: and (a) quasi-periodic orbit for $b = 0.22$, (b) 4-quasi-periodic orbit, for $b = 0.35$, (c) chaotic attractor for $b = 0.44$, (d) chaotic attractor for $b = 0.47$.

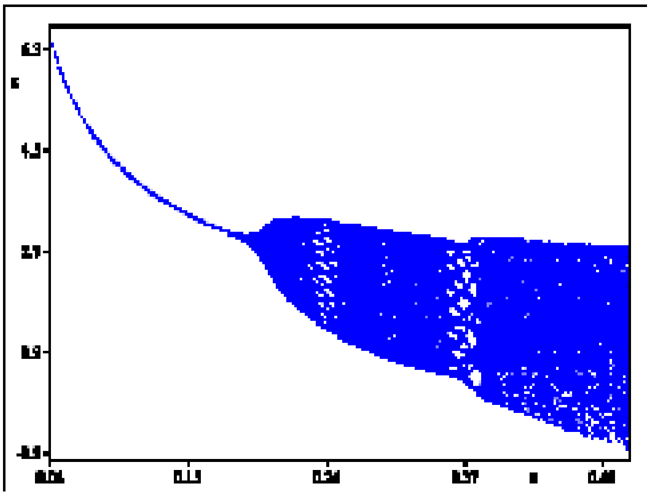

Figure 3: Bifurcation diagram, for the map (2) obtained for $b = 0.3$ and $0.04 \leq a \leq 0.5$.

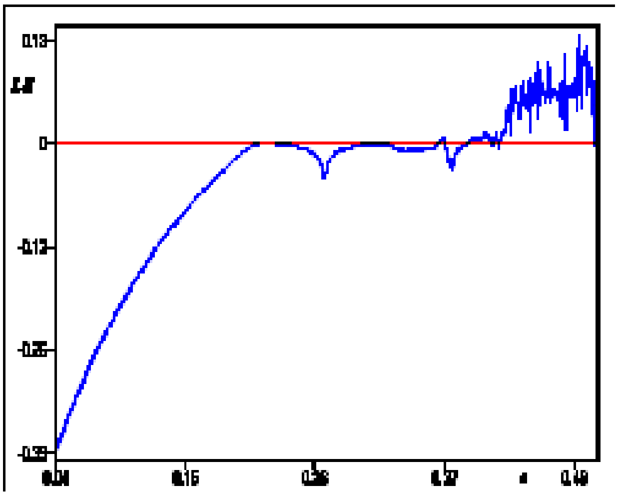

Figure 4: Variation of Lyapunov exponent obtained for $b = 0.3$ and $0.04 \leq a \leq 0.5$.

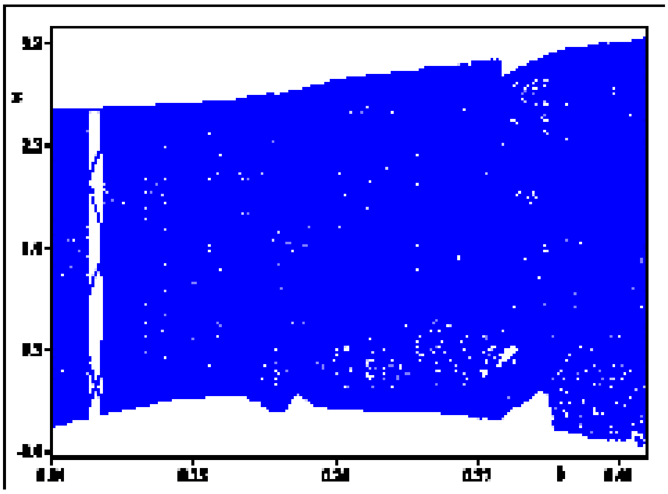

Figure 5: Bifurcation diagram, for the map (2) obtained for $a = 0.4$ and $0.04 \leq b \leq 0.5$.

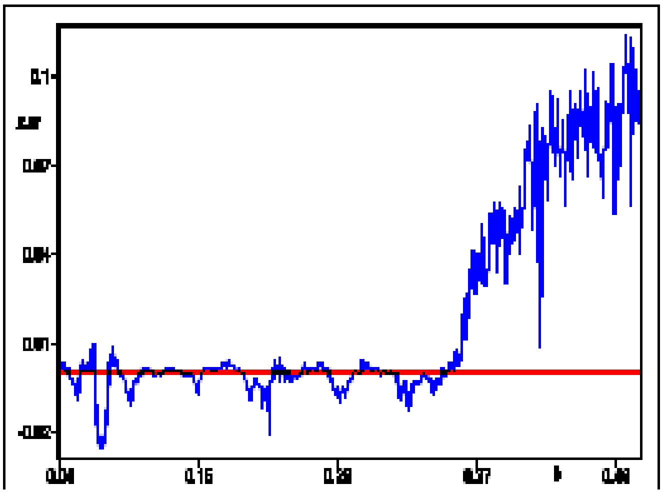

Figure 6: Variation of Lyapunov exponent obtained for $a = 0.4$ and $0.04 \leq b \leq 0.5$.

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