Unsteady Flow of Viscous Fluid through a Straight Long Porous Circular Tube Due to Constant and Exponentially Decaying Pressure Gradient

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Abstract: ***The aim of this paper is to investigate the unsteady flow of viscous fluid through a straight porous channel due to constant and exponentially decaying pressure gradient. The fluid is assumed to be Newtonian and incompressible. To solve the problem we have been using the technique of finite Hankel transformation. The method we have been applied put some restriction to injection parameter that it cannot be ≤ 2ν but in case of suction there is no restriction. In both the cases of constant and exponentially decaying pressure gradient for large oscillation where ν>>1 the amplitude of velocity profile gradually diminishes as suction parameter n increases from '0' to onward which shows that suction tries to annihilate the motion (fig.- 2 and -5) while for injection parameter the amplitude of velocity profile increases when0> n > -2ν (fig.-3 and -6). But when n <2ν the velocity profile become negative and gradually diminishes. Thus n = 2ν can be treated as a suction-injection transition line.***

Keywords: Viscous fluid, Decaying pressure, Navier-Stokes equation, Hankel-Transformation, Suction and injection parameter

1. Introduction

The interest in the flow of viscous Newtonian fluids through porous channel is increasing because of its applications in various discipline like Bio-dynamics, Petroleum engineering, Geo-physics, agricultural fields and so on.

The problem of unsteady flow of viscous incompressible fluid in an annulus of two porous co-axial circular cylinder subjected to suction or injection has been studied by Rao [1] under the presence of a periodic pressure gradient. Singh [2] has studied the flow of viscoelastic Maxwell fluid in the annulus of two porous concentric circular cylinders under the influence of pressure gradient. The problem of flow through straight channel with an arbitrary initial velocity has been consider by *D. Das* and S. Goswami [3]. The problem of flow of a viscous incompressible fluid between two parallel plates one in uniform motion and other at rest with uniform suction at the stationary plate has been solve by Verma and Bansal [4]. D. Das and K. C. Nandy (1992) discussed the motion under gravity of a viscous fluid through a pipe. D. Das and K.C. Nandy (1993) discussed the effect of suction and injection on the flow under the periodic pressure gradient.

In the present paper we have studied the effect of suction and injection on the velocity profile under the constant and exponentially decaying pressure gradient.

2. Formulation of the Problem

In the cylindrical system of co-ordinates (r, θ, z) the Navier-Stokes equations of unsteady motion of viscous incompressible fluid in terms of the velocity components u, v and w along the directions r , θ and z respectively in the absence of external forces are,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}+w\frac{\partial u}{\partial z}-\frac{v}{r^2}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+\upsilon\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}-\frac{u}{r^2}\right) \quad \text{.........} (1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial \theta}+\upsilon\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial z^2}+\frac{1}{r^2}\frac{\partial u}{\partial \theta}-\frac{v}{r}\right) \quad \text{...........}(2)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\upsilon\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial^2 w}{\partial z^2}\right) \quad \text{.............................} (3)$$

Where ρ is the density of fluid and P is the pressure. We now assume that the motion is symmetrical about the z-axis. We have $\frac{\partial}{\partial \theta}=0$ and nature of the motion gives v =0

Applying the above condition the Navier-Stokes equations becomes,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+\upsilon\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right) \quad \text{...........} (4)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\upsilon\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right) \quad ....(5)$$

The equation of continuity is given by,

$$\frac{1}{r}\frac{\partial}{\partial r}(ur)+\frac{\partial w}{\partial z}=0 \quad ....(6)$$

The direction of flow being along the axis of z. This shows that w is independent of z. With this form of velocity and the assumptions that fluid is incompressible. The equation of continuity become simply showing that w is a function of r and t.

Therefore, $\frac{\partial w}{\partial z}=0$ .... (7)

Using the equation (7) the continuity equation (6) become $\frac{1}{r}\frac{\partial}{\partial r}(ur)=0$

Which gives, $ur=-n$ (say) .... (8)

Where $n>0$ is the suction parameter and $n<0$ is the injection parameter.

By the help of equation (8) the equation (4) and (5) reduce to the following forms respectively,

$$\frac{1}{\rho}\frac{\partial P}{\partial r}=\frac{n^2}{r^3} \quad .... (9)$$

$$\frac{\partial w}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\upsilon\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\left(\frac{n}{\upsilon}+1\right)\right] \quad .... (10)$$

Let us substitute,

$$-\frac{1}{\rho}\frac{\partial P}{\partial z}=f(t) \ \& \ \frac{n}{\upsilon}=2k \quad .... (11)$$

Where $f(t)$ is a function of t alone. Therefore the equation (10) become,

$$\frac{1}{\upsilon}\frac{\partial w}{\partial t}=\frac{1}{\upsilon}f(t)+\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}(1+2k)\right] \quad .... (12)$$

Now the initial and boundary conditions are,

w (r, 0) = 0 , at t=0
w(r , t)=0, at r=0, t>0
u=-n/a, at r=a, t>0 .... (13)

Again let,

$$w=Fr^{-k}$$

where F is a function of r and t.

Therefore we get from equation (12),

$$\frac{\partial^2 F}{\partial r^2}+\frac{1}{r}\frac{\partial F}{\partial r}-\frac{k^2}{r^2}F=\frac{1}{\upsilon}\frac{\partial F}{\partial t}-\frac{r^k}{\upsilon}f(t) \quad .... (14)$$

The boundary conditions and initial conditions are,

F(r, t) = 0, at r =a , t >0
F(r, t) = 0 , at $0\le r\le a$, t = 0 .... (15)

3. Solution of the Problem

We introduce the finite Hankel transformation define by,

$$W_H=\int_0^a rwJ_k(r\xi_i)dr \quad .... (16)$$

Where ξ_i are the positive roots of the transcendental equation,

$$J_K(a\xi_1)=0 \quad .... (17)$$

Taking Hankel transform to equation (13) and (14) we get

$$\frac{\partial F_H}{\partial t}+\upsilon\xi_i^2F_H=\frac{a^{k+1}}{\xi_i}J_{k+1}(\xi_i a)f(t)-a\upsilon\xi_i F(a)J_k'(\xi_i a)$$

, k>-1............. (18)

Applying the boundary conditions given by equation (15) the solution of equation (18) is,

$$F_H=\frac{a^{k+1}}{\xi_i}J_{k+1}(a\xi_i)\int_0^t f(\tau)e^{-\upsilon\xi_i^2(t-\tau)}d\tau \quad .... (19)$$

We get from the inversion formula,

$$F(r,t)=\frac{2}{a^2}\sum_i F_H\frac{J_k(r\xi_i)}{[J_{k+1}(a\xi_i)]^2}$$

$$=2a^{k-1}\sum_i\frac{J_k(r\xi_i)}{\xi_i J_{k+1}(a\xi_i)}\int_0^t f(\tau)e^{-\upsilon\xi_i^2(t-\tau)}d\tau$$

Therefore,

$$W(r,t) = 2a^{k-1}r^{-k}\sum_i \frac{J_k(r\xi_i)}{\xi_i J_{k+1}(a\xi_i)}\int_0^t f(\tau)e^{-\upsilon\xi_i^2(t-\tau)}d\tau$$

.... (20)

Where the summation extends over the positive roots of the equation,

$$J_k(a\xi_i) = 0$$

.... (21)

CASE-I: We choose the function**,**

$$f(\tau) = g_0$$

..................(22)

Where g_0 is a constant.

Then, $F(r,t) = 2a^{k-1}g_0\sum_i \frac{J_k(r\xi_i)}{J_{k+1}(a\xi_i)}\int_0^t e^{-\upsilon\xi_i^2(t-\tau)}d\tau$

Therefore

$$W(r,t) = 2a^{k-1}r^{-k}g_0\sum_i \frac{J_k(r\xi_i)}{\xi_i J_{k+1}(a\xi_i)}\int_0^t e^{-\upsilon\xi_i^2(t-\tau)}$$

$$= 2a^{k-1}r^{-k}g_0\sum_i \frac{J_k(r\xi_i)}{\xi_i^3 J_{k+1}(a\xi_i)}\left[\frac{1-e^{-\upsilon\xi_i^2 t}}{\upsilon}\right]$$

...(23)

Using the relation,

$$r^k(1-r^2) = 8(k+1)\sum_i \frac{J_k(r\xi_i)}{\xi_i^3 J_{k+1}(a\xi_i)}$$

.... (24)

We get,

$$W(r,t) = \frac{g_0 a^{k-1}(1-r^2)}{4(k+1)}$$

for k = -1 .... (25) The equation (25) corresponds to one of the most salient feature of this problem is that for large value of t the problem reduce to the Hagen-Poiseuille equation for the steady flow of viscous liquid through a circular pipe due to a constant pressure gradient.

The velocity profile of viscous incompressible fluid given by equation(25) due to constant pressure gradient for different values of k for high viscous oscillation $\upsilon >> 1$ are tabulated in table- I to III and plotted in fig. - 2 to 4.

Table I: Variation of velocity profile for different positive values of k under constant pressure gradient.

r→	0	0.2	0.4	0.6	0.8	1.0
K=0 W→	0.250	0.240	0.210	0.160	0.090	0
K=0.5 W→	0.167	0.160	0.140	0.107	0.060	0
K=1.0 W→	0.125	0.120	0.105	0.080	0.045	0

(a=1, υ=10, t=1, g_0=10)

Table II: Variation of velocity profile for different negative values of k under constant pressure gradient.

r →	0	0.2	0.4	0.6	0.8	1.0
K=-0.1 w →	0.278	0.267	0.233	0.178	0.1	0.0
k=-0.5 w →	0.50	0.48	0.42	0.32	0.18	0.0
K=-0.9 w →	2.50	2.40	2.10	1.60	0.90	0.0

(a=1, υ=10, t=1, g_0=10)

Table III: Variation of velocity profile for different negative values of k under constant pressure gradient.

r →	0.0	0.2	0.4	0.6	0.8	1.0
K=-1.1 w →	-2.50	-2.40	-2.10	-1.60	-0.90	0.0
K=-1.3 w →	-0.833	-0.80	-0.70	-0.533	-0.30	0.0
K=-1.5 w →	-0.50	-0.48	-0.42	-0.32	-0.18	0.0

(a=1, υ=10, t=1, g_0=10)

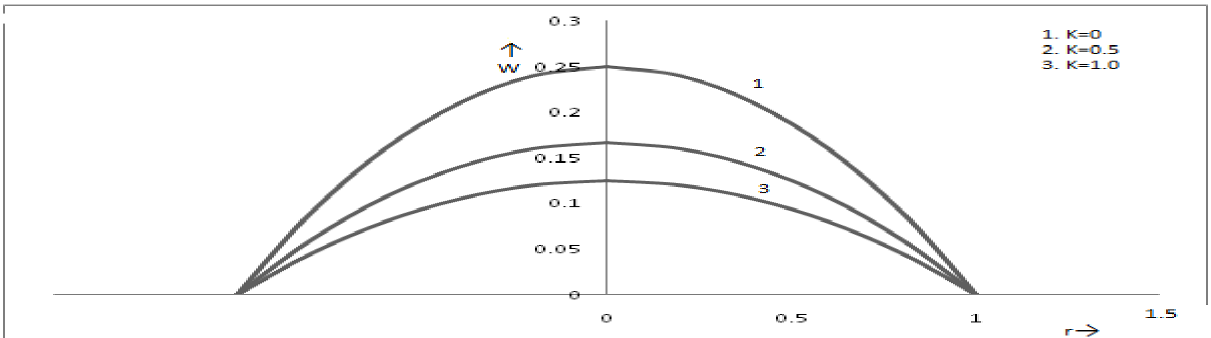

Fig. 2: Variation of velocitry profile for different positive values of K under constant pressure gradient

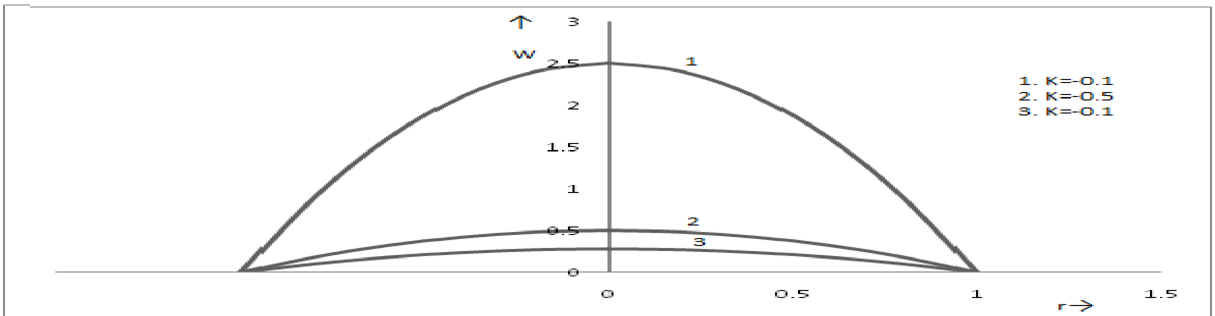

Fig. 3:Variation of velocity profile for different negative values of K under constant pressure gradient.

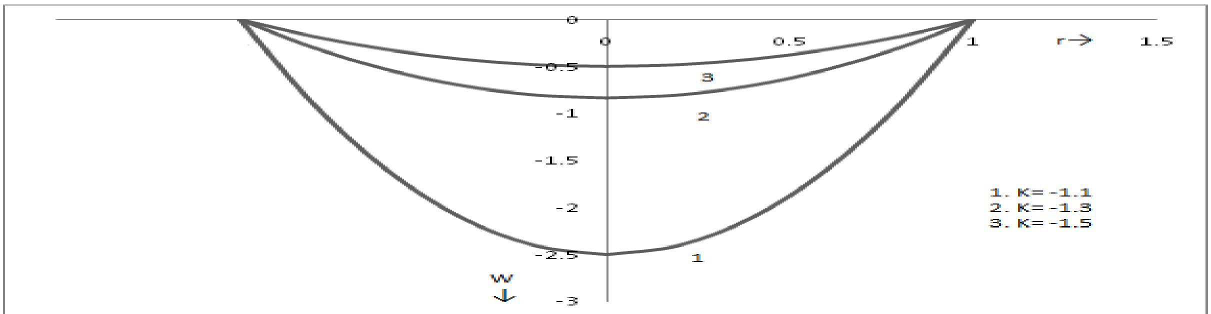

Fig. 4: Variation of velocity profile for different negative values of K under constant pressure gradient

CASE-II: We chose the function

$$f(t) = e^{-\lambda\tau} \quad \text{.......... (26)}$$

Then,

$$F(r,t) = 2a^{k-1}\sum_i \frac{J_k(r\xi_i)}{\xi_i J_{k+1}(a\xi_i)} \frac{e^{-\lambda t} - e^{-\upsilon\xi_i^2 t}}{\upsilon\xi_i^2 - \lambda}$$

Therefore,

$$W(r,t) = 2a^{k-1}r^{-k}\sum_i \frac{J_k(r\xi_i)}{\xi_i J_{k+1}(a\xi_i)} \frac{e^{-\lambda t} - e^{-\upsilon\xi_i^2 t}}{\upsilon\xi_i^2 - \lambda} \quad \text{.......... (27)}$$

Using the relation given by equation (24), we get,

$$W(r,t) = \frac{a^{k-1}(1-r^2)}{4(k+1)} . \frac{e^{-\lambda t} - e^{-\upsilon\xi_i^2 t}}{\upsilon - \frac{\lambda}{\xi_i^2}} \quad \text{.......... (28)}$$

The equation (28) corresponds to the steady flow of viscous liquid through a porous straight channel due to a gradually diminishing pressure gradient.

The velocity profile of viscous incompressible fluid given by equation (28) due to gradually diminishing pressure gradient for different values of k for high viscous oscillation $\upsilon >> 1$ are tabulated in table- IV to VI and plotted in fig. - 5 to 7.

Table IV: Variation of velocity profile for different positive values of k under constant pressure gradient

r $\rightarrow$	0.0	0.2	0.4	0.6	0.8	1.0
K=0 w $\rightarrow$	0.02265	0.02174	0.019026	0.014496	0.00815	0.0
K=0.5 w $\rightarrow$	0.0151	0.014496	0.012684	0.00966	0.005436	0.0
K=1.0 w $\rightarrow$	0.011325	0.0108	0.009513	0.007248	0.004077	0.0

(a=1, υ=10, t=1, $\lambda = 0.1$, $\xi = 2.405$)

Table V: Variation of velocity profile for different negative values of k under constant pressure gradient.

r $\rightarrow$	0.0	0.2	0.4	0.6	0.8	1.0
K=-0.1 w $\rightarrow$	0.02517	0.02416	0.02114	0.016107	0.00906	0.0
K=-0.5 w $\rightarrow$	0.0453	0.04349	0.03805	0.02899	0.00108	0.0
K=-0.9 w $\rightarrow$	0.2265	0.21744	0.19026	0.14496	0.08154	0.0

(a=1, υ=10, t=1, $\lambda = 0.1$, $\xi = 2.405$)

Table VI: Variation of velocity profile for different negative values of k under constant pressure gradient.

r $\rightarrow$	0.0	0.2	0.4	0.6	0.8	1.0
K=-1.1 w $\rightarrow$	-0.2265	-0.2174	-0.1903	-0.1450	-0.0815	0.0
K=-1.3 w $\rightarrow$	-0.0755	-0.0723	-0.0634	-0.0483	-0.0272	0.0
K=-1.5 w $\rightarrow$	-0.0453	-0.0435	-0.0381	-0.0290	-0.0163	0.0

(a=1, υ=10, t=1, $\lambda = 0.1$, $\xi = 2.405$)

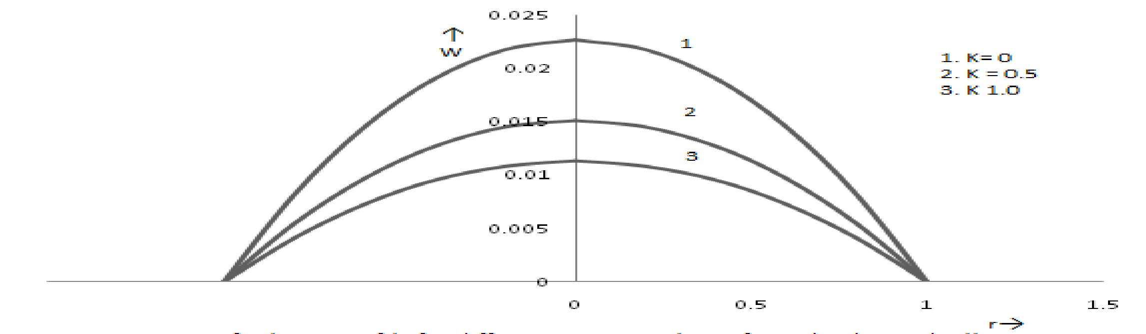

Fig. 5: Variation of velocity profile for different positive values of K under the gradually diminishing pressure gradient.

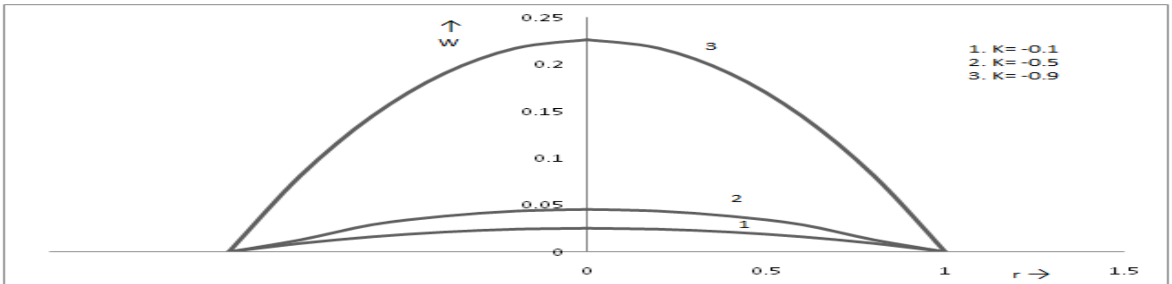

Fig.6:Variation of velocity profile for different negative values of K under the gradually dimishing pressure gradient .

Paper ID: SR201208111651 DOI: 10.21275/SR201208111651

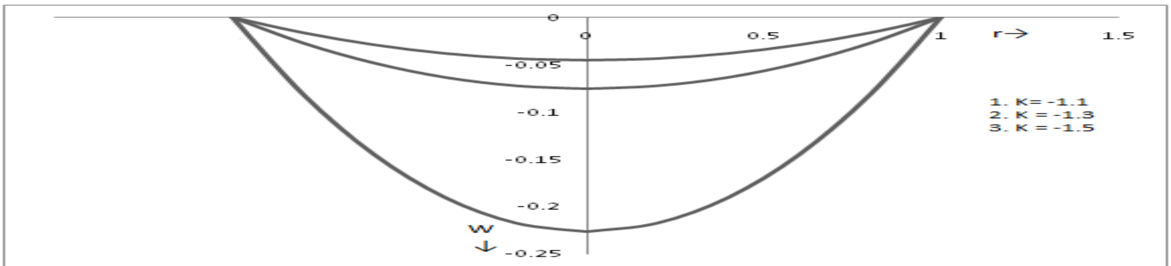

Fig.-7: Variation of velocity profile for differenet negative values of K under the gradually diminishing pressure gradient.

From fig.-2 and -5 it is revealed that the amplitude of velocity profile gradually diminishes as the suction parameter n increase from 0 (zero) to onwards which shows that suction tries to annihilate the motion. Fig.-3 and -6 shows that amplitude of velocity profile increases as injection parameter increases when $0>n>-2\nu$. But when $n<-2\nu$ the amplitude of velocity profile become negative and gradually diminishes. Thus $n = -2\nu$ can be treated as a suction-injection transition line.

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