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Common Fixed Point Theorems under Some Contractive Conditions

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Abstract: ***In this paper, inspired by the concept of some contractive conditions, some new common fixed point theorems under contractive conditions for mapping satisfying a new property is established.***

Keywords: Fixed points, Contractive maps

1. Introduction

Fixed point theorems are statements containing sufficient conditions that ensure the existence of a fixed point. Therefore one of the central concern in fixed point theory is to find a minimal set of sufficient conditions which guarantee a fixed point or a common fixed point. Common fixed point theorems for contractive mapping in a complete metric space, ensure the existence of common fixed point. In 1986, Jungck [4] introduced the notion of compatible maps. Generalization of Jungck's contraction conditions have been extensively used to study common fixed point theory of contractive mappings. However, fixed point theory for non compatible mappings is equally interesting. Pant [5,6] initiated some work along these lines. The main aim of this paper is to give a common fixed point theorem under some contractive conditions.

2. Preliminaries

Definition [2] Let X be a non empty set. A mapping $d: X \times X \to R$ (The set of reals) is said to be a metric or distance function if d satisfying the following axioms. For all x, y, z∈ X

1) $d(x,y) \geq 0$,
2) $d(x,y) = 0 \text{ iff } x = y$,
3) $d(x,y) = d(y,x)$,
4) $d(x,y) \leq d(x,z) + d(z,y)$.

If d is a metric for X, then the ordered pair (X,d) is called a metric space and $d(x,y)$ is called the distance between x and y.

Example [2] Let X be an arbitrary non empty set and $d: X \times X \to R$ be a function such that

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

Then (X,d) is a metric space

Definition [2] A sequence in a metric space is a Cauchy sequence if for every $\varepsilon > 0$ ∃ n0∈N such that $d(x_n, x_m) < \varepsilon, \forall\, n, m > n_0$.

Example In a metric space (0, 1] with usual metric $d(x,y) = |x-y|$, the sequence $a_n = \frac{1}{n}$ is a Cauchy sequence.

Example let us consider the space Q of rational number with usual metric $d(x,y) = |x-y|$, then the sequence ⟨1.4,1.41,1.414,1.4142,----⟩ of finite decimal is a Cauchy sequence in Q.

Definition [2] A metric space (X,d) is said to be complete if every Cauchy sequence of points in X converge to a point in X.

Example The usual metric space (R,d) is a complete metric space.

Example The space of complex numbers is a complete metric space.

Definition [2] Let (X,d) be a metric space. A mapping $f: X \to X$ is called a contraction mapping or principle if there exists a real number α with $0 \leq \alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x,y) < d(x,y)\ \forall\, x, y \in X$.

Thus in a contraction on X, the distance between the images of any two points is less than the distance between the points.

Hence the application of 'f' to each of two points 'contracts' the distance between them

Definition [7]. Two self mappings U and V of a metric space (X,d) are said to be weakly commuting if

$$d(UVx, VUx) \leq d(Vx, Ux), \forall x \in X.$$

Definition [4]. Let U and V be two selfmappings of a metric space (X,d). U and V are said to be compatible if

$$\lim_{n\to\infty} d(VUx_n, UVx_n = 0)$$

Whenever (x_n) is a sequence in X such that

$$\lim_{n\to\infty} Vx = \lim_{n\to\infty} Ux = t\ n\to\infty$$

For some $t \in X$.

Definition [4]. Two self mappings U and V of a metric space(X, d) are said to be weakly compatible if they commute at their coincidence points; that is

$Ux = Vx$ *for some* $x \in X$, *then* $UV \langle = VUx$.

Main Result

Definition [1] Let U and V be two self mappings of a metric space (X, d). We say that U *and* V satisfy the property $(E.A)$ if there exists a sequence (x_n) such that

$$\lim_{n\to\infty} Ux = \lim_{n\to\infty} Vx = t$$

For some $t \in$ ~.

Example Let $X = [0, +\infty]$. Define $U, V : X \to X$ by

$$Ux = \frac{x}{5} \text{ and } Vx = \frac{3x}{5}, \ \forall x \in X.$$

Consider the sequence $x_n = \frac{1}{n}$

Clearly $\lim_{n\to\infty} Ux_n = \lim_{n\to\infty} Vx_n = 0$

Then U *and* V satisfy (E.A).

Theorem [2] Every closed subspace of a complete metric space is complete.

Proof Let S be a closed subspace of a complete metric space X. Let (x_n) be a Cauchy sequence in S. Then (x_n) is a Cauchy sequence in X and hence it must converge to a point x *in* X.

But then $x \in \overline{S} = S$. Thus S is complete.

Conversely

Let S be a complete subspace of a metric space X. Let $x \overline{\in} X$. Then there is a sequence (x_n) *in* S which converge to x *in* X. Hence (x_n) is a Cauchy sequence in S. Since S is complete, (x_n) must converge to some point, say, y *in* S. By uniqueness of limit, we must have $x = y \in S$.

Hence $\overline{S} = S$, that is S is closed.

Theorem *Let* U *and* V *be two weakly compatible self mappings of a complete metric space (X, d) such that*

1) *satisfy the property (E.A),*
2) $d(Vx,Vy) < max \ \{d(Sx,Sy),[d(Vx,Ux)+d(Vy,Uy)]/2, [d(Vy, Ux) + d(Vx,Uy)] /2\}, \ \forall x \neq y \in X, \in$
3) $VX \subset UX$

If UX *or* VX is a closed subspace of X, then T *and* S have a unique comman fixed point.

Proof : Since V and U satisfy the property (E.A), there exists a sequence (x_n) in X satisfying

$$\lim_{n\to\infty} Vx = \lim_{n\to\infty} Ux = t \text{ for some } t \in X.$$

Suppose that UX is closed. Since every closed subset of a complete metric space is complete. So UX is also complete.

Then $\lim_{n\to\infty} Ux = Ua$ for some $a \in X$.

Also, $\lim_{n\to\infty} Vx = Ua$.

We show that$Va = Ua$. We prove it by contradiction. Let us suppose that $Va \neq Ua$.

Condition (ii) imply that

$d(Vx_n, Va) < max \ \{d(Ux_n,Ua), [d(Vx_n,Ux_n) + d(Va,Ua)]/2, [d(Va,Ux_n) + d(Vx_n,Ua)]/2\}$

Letting $n \to \infty$, implies

$d(Ua,Va) \leq max \ [d(Ua,Ua),[d(Va,Ua) + d \ (Ua,Ua)/2], [d(Va,Ua) + d(Ua,Ua)] \ 2] \leq d \ (Va,Ua)/2.$

A contradiction. Hence $Va = Ua$.

Since V $\neg nd$ U *are weakly comptible*, $UVa = VUa$ *and therefore*, $VVa = VUa = UVa = UUa$.

Finally, we show that Va *is comman fixed point of* V*and* U. *Suppose that* $Va \neq V$ 搜a. *Then*

$d\ (Va, VVa)$
$\leq max \ \{d(Ua,UVa),[d(Va,Ua) + d(VVa,UVa)]/2, [d(VVa, Ua) + d(Va,UVa)]/2\})$

$\leq max\{d(Va, VVa), d(VVa, Ta)\} = d(Va, VVa)$.

This is a contradiction

Hence $VVa = Va$ *and* $UVa =$ ⍰$Va = Va$. *The proof is similar* When VX is assumed to be a closed subspace of X. Since $VX \subset UX$. Uniqueness of the common fixed point follows easily.

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